

# AFFINE HECKE ALGEBRAS AND QUANTUM SYMMETRIC PAIRS

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**ABSTRACT.** We introduce an affine Schur algebra via the affine Hecke algebra associated to Weyl group of affine type C. We establish multiplication formulas on the affine Hecke algebra and affine Schur algebra. Then we construct monomial bases and canonical bases for the affine Schur algebra. The multiplication formula allows us to establish a stabilization property of the family of affine Schur algebras that leads to the modified version of an algebra  $\mathbf{K}_n^c$ . We show that  $\mathbf{K}_n^c$  is a coideal subalgebra of quantum affine algebra  $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ , and  $(\mathbf{U}(\widehat{\mathfrak{gl}}_n), \mathbf{K}_n^c)$  forms a quantum symmetric pair. The modified coideal algebra is shown to admit monomial and stably canonical bases. We also formulate several variants of the affine Schur algebra and the (modified) coideal algebra above, as well as their monomial and canonical bases. This work provides a new and algebraic approach which complements and sheds new light on our previous geometric approach on the subject.

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## 1. INTRODUCTION

**1.1. History.** Dipper and James [DJ89] introduced the finite Schur algebra as the endomorphism algebra of a sum of permutation modules of the finite type A Hecke algebra. (All Schur algebras in this paper are understood as quantum Schur algebras.) Around the same time, the same Schur algebra was constructed geometrically in [BLM90]. The paper [BLM90] further provides a construction of the (modified) quantum group of finite type A and its stably canonical basis by studying the stabilization property of the structures of the family of the Schur algebras.

There have been several works on the generalization to affine type A of the Schur algebras [GV93, Gr99, Lu99, SV00, DDF12]. However the stabilization phenomenon in the affine type A setting is understood only in recent years [DF14, DF15]. In the affine type A, the Chevalley generators do not form a generating set for the affine Schur algebras or the corresponding stabilization algebra. It was shown in [DF14] (also cf. [FL15]) that there exists

a generating set consisting of semisimple generators which correspond to the bidiagonal matrices; moreover a multiplication formula with semisimple generators was provided. A different approach for some main results in [DF14] was given in [FLLLW].

The Schur algebra via the finite type B/C Hecke algebra à la Dipper-James was studied by R. Green [Gr97] (also cf. [HL06]). This construction was generalized and put in a much broader context under the so-called  $\iota$ -Schur duality [BW13], where the coideal subalgebra of the quantum group of type A features naturally. A geometric realization of such a Schur algebra via flag varieties of type B/C is given in [BKLW] (and also [LW15]), where a BLM-type stabilization leads to a (modified) coideal subalgebra of the quantum group of type A and its canonical basis. An affinization of this geometric construction in the setting of affine type C flag varieties is developed extensively in [FLLLW].

**1.2. Goal.** This is a companion paper of our previous work [FLLLW]. In this paper we shall develop a Hecke algebraic approach toward the affine Schur algebras, their stabilization algebras, as well as their monomial and canonical bases. This approach provides new algebraic constructions and proofs which shed new light on the underlying algebra structures, and it complements the geometric constructions in [FLLLW]. We further identify the algebras constructed here algebraically with the corresponding algebras constructed geometrically in *loc. cit.*. In this Hecke algebraic framework, we construct a quantum symmetric pair consisting of the (non-modified) quantum affine  $\mathfrak{gl}_n$  and its coideal subalgebra.

Let us explain our approach and results in detail.

### 1.3. Main results.

**1.3.1. Multiplication formulas.** Let  $\mathbf{H}$  be the Hecke algebra associated to the Weyl group  $W = W(d)$  of affine flag variety of type C, with generators  $T_w$  for  $w \in W$ . We caution that while the algebra  $\mathbf{H}$  is naturally associated to affine flag variety of type C, it is often viewed in literature from Langlands dual viewpoint, and referred as affine Hecke algebra of type B.

Following Lusztig's presentation in affine type A, there have been two presentations of affine Weyl groups of type C as permutation groups on  $\mathbb{Z}$ ; cf. [B86, Shi94, EE98, BB05], and their length function formulas are given in [EE98, BB05]; see (2.1.9). In this paper, we choose the presentation of  $W$  as a permutation group on  $\mathbb{Z}$  with two fixed points in each period, which makes the symmetries of  $W$  more transparent (cf. [Shi94]). In particular, this leads to a new and simple length function formula; see Lemma 2.1.

We define the affine Schur algebra  $\mathbf{S}_{n,d}^c$  as the endomorphism algebra of a sum of permutation modules of the affine Hecke algebra  $\mathbf{H}$ . Like in [DF14, DF15], we shall develop a new multiplication formula for the affine Schur algebra  $\mathbf{S}_{n,d}^c$ . However there is a major difference here from affine type A. The semisimple generators in *loc. cit.* are bar invariant (and correspond to closed orbits geometrically [FL15]), and the multiplication formula therein does not require nontrivial multiplication on the Hecke algebra level. In contrast, we shall see that the generators in our setting correspond to tridiagonal matrices, and they are not bar invariant in general (neither do they correspond to closed orbits on partial affine flag varieties [FLLLW]).

To develop a multiplication formula for affine Schur algebra, we are led to first establish a multiplication formula at the affine Hecke algebra level, which is technically challenging and combinatorially involved. For notations we refer to the sentence above Theorem 3.3.

**Theorem A** (Theorem 3.3). *Let  $B = \kappa(\lambda, g_1, \mu)$  be a tridiagonal matrix. Then, for any  $g_2 \in \mathcal{D}_{\mu\nu}$  and  $w \in \mathcal{D}_{\delta(B)} \cap W_\mu$ , we have*

$$T_{g_1} T_{wg_2} = \sum_{\sigma \in K_w} (v^2 - 1)^{n(\sigma)} v^{2h(w, \sigma)} T_{g_1 \sigma w g_2}.$$

By construction, the affine Schur algebra  $\mathbf{S}_{n,d}^c$  admits a basis  $\{e_A \mid A \in \Xi_{n,d}\}$  parametrized by the set  $\Xi_{n,d}$  (2.3.5) of  $n$ -periodic centro-symmetric  $\mathbb{Z} \times \mathbb{Z}$  matrices over  $\mathbb{N}$  of size  $d$ . (We sometimes normalize  $e_A$  to become the standard basis  $[A]$ .) We have the following multiplication formula for the affine Schur algebra  $\mathbf{S}_{n,d}^c$ , and shall refer to the paragraph above Theorem 4.11 for notations.

**Theorem B** (Theorems 4.11 and 5.6). *Let  $A, B \in \Xi_{n,d}$  with  $B$  tridiagonal and  $\text{row}_c(A) = \text{col}_c(B)$ . Then we have*

$$e_B e_A = \sum_{\substack{T \in \Theta_{B,A} \\ S \in \Gamma_T}} (v^2 - 1)^{n(S)} v^{2(\ell(A, B, S, T) - n(S) - h(S, T))} \llbracket A; S; T \rrbracket e_{A(T-S)}.$$

Multiplication formulas in different form can also be derived from the geometric approach in [FL16]. A special case of Theorem B gives a formula which closely resembles the multiplication formula in affine type A in [DF14], and another special case closely resembles the multiplication formula in type B/C in [BKLW]. While the multiplication formula in Theorem B is explicit yet complicated, one can read off useful and essential information about the algebra  $\mathbf{S}_{n,d}^c$ . For example, it allows us to show that the algebra  $\mathbf{S}_{n,d}^c$  has a generating set given by  $[A]$  for tridiagonal matrices  $A \in \Xi_{n,d}$ ; see Theorem 5.16(c).

**1.3.2. Monomial and canonical bases.** By developing further multiplicative properties of  $\mathbf{S}_{n,d}^c$  from the multiplication formula in Theorem B, we produce an algorithm (see Algorithm 5.15) to construct a monomial basis  $\{m_A \mid A \in \Xi_{n,d}\}$  which is bar invariant such that  $m_A = [A] +$  lower terms with respect to a natural partial ordering. (A version of this algorithm produces a monomial basis in affine type A [LL15].) The construction of the canonical basis of  $\mathbf{S}_{n,d}^c$  via the Kazhdan-Lusztig basis of  $\mathbf{H}$  is rather routine (cf. [Du92]).

**Theorem C** (Theorem 5.8, Theorem 5.18). *The Schur algebra  $\mathbf{S}_{n,d}^c$  admits both monomial and canonical bases.*

The multiplication formula in Theorem B also allows us to establish a stabilization property of the family of algebras  $\mathbf{S}_{n,d}^c$  as  $d \mapsto \infty$ . While largely following the strategy of [BLM90] (which has been applied also to [BKLW, DF15]), our setting is technically a little more involved. The stabilization property leads to the construction of a stabilization algebra  $\dot{\mathbf{K}}_n^c$ . A variant of the multiplication formula in Theorem B is valid for the algebra  $\dot{\mathbf{K}}_n^c$ ; see Theorem 6.4. The algorithm for monomial basis for the affine Schur algebra  $\mathbf{S}_{n,d}^c$  is adapted here to construct a monomial basis for the stabilization algebra  $\dot{\mathbf{K}}_n^c$ . The stably canonical basis for  $\dot{\mathbf{K}}_n^c$  follows from the existence of its monomial basis.

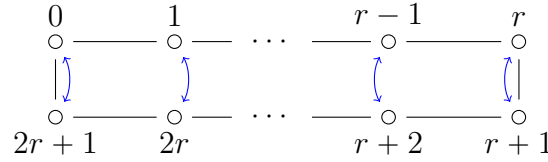
**Theorem D** (Theorem 6.7). *We have an algebra  $\dot{\mathbf{K}}_n^c$  arising from stabilization on the family of Schur algebras  $\mathbf{S}_{n,d}^c$  (as  $d$  varies). Moreover,  $\dot{\mathbf{K}}_n^c$  admits both monomial and stably canonical bases.*

We emphasize that the constructions in Section 3 throughout Section 7 are entirely independent of the geometric approach developed in [FLLLW]. Our constructions and proofs for Theorems C and D are built on the multiplication formulas in Theorems A and B, while Theorems A and B are new. We show in Propositions 2.14 and 6.8 that the algebras  $\mathbf{S}_{n,d}^c$  and  $\dot{\mathbf{K}}_n^c$  here are isomorphic to the geometrically constructed algebras in the same notation in [FLLLW], and their monomial and canonical bases also match with their counterparts in *loc. cit.*.

**1.3.3. Affine quantum symmetric pairs.** Recall a quantum symmetric pair  $(\mathbf{U}, \mathbf{U}^e)$  in the sense of [Le02, Ko14] consists of a quantum group  $\mathbf{U}$  and its coideal subalgebra  $\mathbf{U}^e$ . The stabilization algebra for Schur algebras of finite type B/C is the (modified) coideal subalgebra of the quantum group of finite type A [BKLW, LW15]. There is a standard procedure (which goes back to [BLM90]) to construct an algebra  $\mathbf{K}_n^c$  for which  $\dot{\mathbf{K}}_n^c$  is the modified (or idempotent) version. Similarly, in the affine type A setting we have a stabilization algebra  $\dot{\mathbf{K}}_n$  which is the modified version of  $\mathbf{K}_n$ , and moreover,  $\mathbf{K}_n$  is isomorphic to the quantum affine  $\mathfrak{gl}_n$ ; cf. [DF15].

**Theorem E** (Theorem 7.8). *The pair  $(\mathbf{K}_n, \mathbf{K}_n^c)$  forms a quantum symmetric pair associated to an involution on the Dynkin diagram of affine type A depicted in Figure 1.*

FIGURE 1. Dynkin diagram of type  $A_{2r+1}^{(1)}$  with involution of type  $jj \equiv c$ .



Theorem E does not appear explicitly in the framework of [FLLLW] (only an idempotent quantum symmetric pair was established therein). The passage from the idempotent quantum symmetric pair to the quantum symmetric pair statement here is nontrivial, and its proof takes advantage of some constructions in *loc. cit.* together with the multiplication formula for  $\dot{\mathbf{K}}_n^c$  in Theorem B.

We remark that in a very interesting paper [CGM14] Chen, Guay and Ma also studied interactions between affine Hecke algebra and quantum symmetric pair from a very different perspective from ours.

**1.3.4. Variants.** There are several variants (called type  $ij$ ,  $jn$ , and  $in$ ) of the algebras  $\mathbf{S}_{n,d}^c$  and  $\dot{\mathbf{K}}_n^c$  (here we can view  $c = jj$ ). The Schur algebras  $\mathbf{S}_{n,d}^{ij}$ ,  $\mathbf{S}_{n,d}^{jn}$ ,  $\mathbf{S}_{\eta,d}^{in}$  are subalgebras of  $\mathbf{S}_{n,d}^c$  by construction. We also construct their respective stabilization algebras  $\dot{\mathbf{K}}_n^{ij}$ ,  $\dot{\mathbf{K}}_n^{jn}$ ,  $\dot{\mathbf{K}}_{\eta}^{in}$  together with their canonical bases.

**Theorem F** (Theorem 8.6, Theorem 8.14, Remark 8.15). *The Schur algebra  $\mathbf{S}_{n,d}^{ij}$  admits a canonical basis compatible with the one in  $\mathbf{S}_{n,d}^c$  under the inclusion  $\mathbf{S}_{n,d}^{ij} \subset \mathbf{S}_{n,d}^c$ . The algebra,*

$\dot{\mathbf{K}}_n^u$  is isomorphic to a subquotient of  $\dot{\mathbf{K}}_n^c$ , with compatible standard, monomial, and stably canonical bases.

Similar results for types  $j$  and  $u$  also hold and can be found in Theorem 8.19 and Theorem 8.22.

**1.4. The organization.** This paper is organized as follows. In Section 2, the affine Schur algebra  $\mathbf{S}_{n,d}^c$  is defined via the affine Hecke algebra  $\mathbf{H}$ , and then identified with the geometrically-defined affine Schur algebra from [FLLW] with compatible standard bases.

In Section 3, a multiplication formula for affine Hecke algebra  $\mathbf{H}$  is formulated.

In Section 4, we establish a key multiplication formula for the affine Schur algebra  $\mathbf{S}_{n,d}^c$  with tridiagonal generators.

In Section 5, canonical basis for the affine Schur algebra  $\mathbf{S}_{n,d}^c$  is constructed and identified with the one defined geometrically in [FLLW]. Using the multiplication formula in Section 4, we construct a monomial basis for  $\mathbf{S}_{n,d}^c$ .

In Section 6, we shall establish a stabilization property for the family of affine Schur algebras  $\mathbf{S}_{n,d}^c$  as  $d$  varies, which leads to a quantum algebra  $\dot{\mathbf{K}}_n^c$ . A monomial basis and a stably canonical basis for  $\dot{\mathbf{K}}_n^c$  are constructed.

In Section 7, we construct an algebra  $\mathbf{K}_n^c$  for which  $\dot{\mathbf{K}}_n^c$  is the modified (idempotentized) version. We show that  $(\mathbf{K}_n, \mathbf{K}_n^c)$  forms a quantum symmetric pair.

In Section 8, three more variants of affine Schur algebras and their corresponding stabilization algebras are introduced. We establish various results for these new variants analogous to those for the algebras  $\mathbf{S}_{n,d}^c$  and  $\dot{\mathbf{K}}_n^c$  obtained in earlier sections.

**Notations.** We shall denote  $\mathbb{N} = \{0, 1, 2, \dots\}$ . For  $a, b \in \mathbb{Z}$ , we let

$$[a..b] = [a, b] \cap \mathbb{Z}, \quad [a..b) = [a, b) \cap \mathbb{Z}$$

be the integer intervals. We define similarly integer intervals  $(a..b]$  and  $(a..b)$ .

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## 2. AFFINE SCHUR ALGEBRAS VIA AFFINE HECKE ALGEBRAS

In this section, we define the affine Schur algebra  $\mathbf{S}_{n,d}^c$  as the endomorphism algebra of a sum of permutation modules of the affine Hecke algebra associated to the affine Weyl group of type  $\tilde{C}_d$ , and show that its basis is parametrized by periodic and centro-symmetric integer matrices. We then reformulate some combinatorics of the affine Weyl group and associated Hecke algebra in terms of such integer matrices. The algebra  $\mathbf{S}_{n,d}^c$  is identified with the geometrically-defined affine Schur algebra from [FLLW].

**2.1. Affine Weyl groups.** Let  $r, d \in \mathbb{N}$  be such that  $d \geq 2$ , and set

$$n = 2r + 2, \quad D = 2d + 2. \quad (2.1.1)$$

Let  $W$  be the Weyl group of type  $\tilde{C}_d$  generated by  $S = \{s_0, s_1, \dots, s_d\}$  with the affine Dynkin diagram

$$\begin{array}{ccccccc} \circ & \Longrightarrow & \circ & \text{---} & \cdots & \text{---} & \circ & \Longleftarrow & \circ \\ 0 & & 1 & & & & d-1 & & d \end{array}$$

Then  $(W, S)$  is a Coxeter group. We denote the identity of  $W$  by  $\mathbb{1}$ . It is known (c.f. [B86]) that  $W$  can be identified as a subgroup of the permutation group  $\text{Perm}(\mathbb{Z})$  satisfying certain natural conditions (such a description for Weyl groups of affine type  $A$  goes back to Lusztig). For our purpose, we shall introduce a variant of such a description, which has been given in [Shi94]. We identify  $W$  with the subgroup  $\text{Perm}^c(\mathbb{Z})$  which consists of  $g \in \text{Perm}(\mathbb{Z})$  such that

$$g(i + D) = g(i) + D, g(-i) = -g(i) \quad \text{for } i \in \mathbb{Z}. \quad (2.1.2)$$

In particular, we always have

$$g(0) = 0 \quad \text{and} \quad g(d + 1) = d + 1.$$

Any element  $w$  in  $W \equiv \text{Perm}^c(\mathbb{Z})$  is uniquely determined by its value on  $\{1, 2, \dots, d\}$ , and we shall denote

$$w = \left( \begin{array}{cccc} 1 & 2 & \cdots & d \\ a_1 & a_2 & \cdots & a_d \end{array} \right)_c = [a_1, a_2, \dots, a_d]_c \quad (2.1.3)$$

to mean that  $w(i) = a_i$  for  $1 \leq i \leq d$ . We define the *transposition*

$$(i, j)_c \in W, \quad \text{for } i \neq j, \quad (2.1.4)$$

as the element which swaps  $kD \pm i$  and  $kD \pm j$  ( $k \in \mathbb{Z}$ ) while fixing  $\mathbb{Z} \setminus \{kD \pm i, kD \pm j \mid k \in \mathbb{Z}\}$  pointwise.

Let us establish an explicit isomorphism between  $W$  and the Weyl group of type  $\tilde{C}_d$  denoted by  $\tilde{S}_d^C$  [BB05, §8.4]. (Roughly speaking, the difference is that the permutations in  $W$  fix  $d + 1 + D\mathbb{Z}$ .) The identification  $\tilde{S}_d^C \rightarrow W$  is given by

$$g' \mapsto g = [\iota(g'(1)), \dots, \iota(g'(d))]_c, \quad (2.1.5)$$

where

$$\iota : \mathbb{Z} \longrightarrow \mathbb{Z} \setminus \{d + 1 + D\mathbb{Z}\}, \quad i \mapsto i + \left\lfloor \frac{i - d}{D - 1} \right\rfloor \quad (2.1.6)$$

is an order-preserving bijection. Here the floor and ceiling functions are defined as usual by  $\lfloor a \rfloor = \max\{m \in \mathbb{Z} \mid m \leq a\}$  and  $\lceil a \rceil = \min\{m \in \mathbb{Z} \mid m \geq a\}$  for  $a \in \mathbb{R}$ . In other words, we have

$$\iota(g'(i + k(D - 1))) = g(i + kD), \quad -d \leq i \leq d, k \in \mathbb{Z}. \quad (2.1.7)$$

This identification shows  $W$  is indeed the Weyl group of type  $\tilde{C}_d$ .

We denote by  $|X|$  the cardinality of a finite set  $X$ . The length function  $\ell(\cdot)$  on  $W$  affords the following simple formula (compare with the more involved formula in [BB05, (8.44)], which is recalled in (2.1.9) below).

**Lemma 2.1.** *The length of  $g \in W$  is given by*

$$\ell(g) = \frac{1}{2} \left| \left\{ (i, j) \in [1..d] \times \mathbb{Z} \mid \begin{matrix} i > j \\ g(i) < g(j) \end{matrix} \text{ or } \begin{matrix} i < j \\ g(i) > g(j) \end{matrix} \right\} \right|. \quad (2.1.8)$$

*Proof.* Let  $g' \in \tilde{S}_d^C$  be the element identified with  $g$ . It is known [BB05, (8.44), (8.45)] that

$$\ell(g') = \text{inv}_B(g'(1), \dots, g'(d)) + \sum_{1 \leq i \leq j \leq d} \left( \left\lfloor \frac{|g'(i) - g'(j)|}{D-1} \right\rfloor + \left\lfloor \frac{|g'(i) + g'(j)|}{D-1} \right\rfloor \right). \quad (2.1.9)$$

Since  $\iota$  is order preserving, we have  $g'(i) < g'(j) \Leftrightarrow g(i) < g(j)$  for all  $i, j \in [1..d]$ . Hence by [BB05, (8.2)] we have

$$\begin{aligned} \text{inv}_B(g'(1), \dots, g'(d)) &= \left| \{(i, j) \in [1..d]^2 \mid \begin{matrix} i < j \\ g'(i) > g'(j) \end{matrix}\} \right| + \left| \{(i, j) \in [1..d]^2 \mid \begin{matrix} i \leq j \\ g'(-i) > g'(j) \end{matrix}\} \right| \\ &= \left| \{(i, j) \in [1..d]^2 \mid \begin{matrix} i < j \\ g(i) > g(j) \end{matrix}\} \right| + \left| \{(i, j) \in [1..d]^2 \mid \begin{matrix} i \leq j \\ g(-i) > g(j) \end{matrix}\} \right| \\ &= \frac{1}{2} \left| \{(i, j) \in [1..d] \times [-d..d] \mid \begin{matrix} i > j \\ g(i) < g(j) \end{matrix} \text{ or } \begin{matrix} i < j \\ g(i) > g(j) \end{matrix}\} \right|. \end{aligned}$$

By (2.1.6)–(2.1.7), we obtain that

$$\left\lfloor \frac{|g'(i) \pm g'(j)|}{D-1} \right\rfloor = \left\lfloor \frac{|g(i) \pm g(j)|}{D} \right\rfloor \quad \text{for } i, j \in [1..d].$$

We regard  $W \subset \tilde{S}_D$ , the Weyl group of affine type A in [BB05, §8.3]. [BB05, (8.31)] can be rephrased using  $g$  (instead of  $g'$ ) as follows: for  $g \in W \subset \tilde{S}_D$  and for  $i, j \in [1..d]$ , we have

$$\left\lfloor \frac{|g(i) - g(j)|}{D} \right\rfloor = |\{k \in \mathbb{Z} \mid g(j) > g(i + k(D))\}| + |\{k \in \mathbb{Z} \mid g(i) > g(j + k(D))\}|.$$

A detailed calculation shows that

$$\begin{aligned} \sum_{1 \leq i \leq j \leq d} \left\lfloor \frac{|g(i) + g(j)|}{D} \right\rfloor &= \sum_{1 \leq i < j \leq d} (|\{k \geq 1 \mid g(j) > g(-i + kD)\}| + |\{k \geq 1 \mid g(i) < g(-j - kD)\}|) \\ &\quad + \sum_{1 \leq i \leq d} (|\{k \geq 1 \mid g(i) > g(-i + kD)\}| + |\{k \geq 1 \mid g(i) < g(-i - kD)\}|) \\ &= \frac{1}{2} \sum_{k=1}^{\infty} \left( \begin{aligned} &|\{(j, -i + kD) \mid 1 \leq i < j \leq d, g(j) > g(-i + kD)\}| \\ &+ |\{(i, -j + kD) \mid 1 \leq i < j \leq d, g(i) > g(-j + kD)\}| \\ &+ |\{(i, -j - kD) \mid 1 \leq i < j \leq d, g(i) < g(-j - kD)\}| \\ &+ |\{(j, -i - kD) \mid 1 \leq i < j \leq d, g(j) < g(-i - kD)\}| \\ &+ |\{(i, -i + kD) \mid 1 \leq i \leq d, g(i) > g(-i + kD)\}| \\ &+ |\{(i, \frac{kD}{2}) \mid 1 \leq i \leq d, g(i) > g(\frac{kD}{2})\}| \\ &+ |\{(i, -i - kD) \mid 1 \leq i \leq d, g(i) < g(-i - kD)\}| \\ &+ |\{(i, -\frac{kD}{2}) \mid 1 \leq i \leq d, g(i) < g(-\frac{kD}{2})\}| \end{aligned} \right). \end{aligned}$$

Similarly we have

$$\begin{aligned}
& \sum_{1 \leq i \leq j \leq d} \left\lfloor \frac{|g(i) - g(j)|}{D} \right\rfloor \\
&= \sum_{1 \leq i < j \leq d} (|\{k \geq 1 \mid g(i) > g(j + kD)\}| + |\{k \geq 1 \mid g(j) > g(i + kD)\}|) \\
&= \frac{1}{2} \sum_{k=1}^{\infty} \left( \begin{aligned} & |\{(i, j + kD) \mid 1 \leq i < j \leq d, g(i) > g(j + kD)\}| \\ & + |\{(j, i - kD) \mid 1 \leq i < j \leq d, g(i - kD) > g(j)\}| \\ & + |\{(j, i + kD) \mid 1 \leq i < j \leq d, g(j) > g(i + kD)\}| \\ & + |\{(i, j - kD) \mid 1 \leq i < j \leq d, g(j - kD) > g(i)\}| \end{aligned} \right).
\end{aligned}$$

The lemma then follows by simplifying the summation of the relevant terms above.  $\square$

Via the notation (2.1.3), the generators of  $W$  are given by

$$\begin{aligned}
s_0 &= [-1, 2, 3, \dots, d-1, d]_c, \\
s_d &= [1, 2, 3, \dots, d-1, d+2]_c, \\
s_i &= [1, \dots, i-1, i+1, i, i+2, \dots, d]_c = (i, i+1)_c, \quad \text{for } i = 1, \dots, d-1.
\end{aligned} \tag{2.1.10}$$

**2.2. Parabolic subgroups and cosets.** Denote the set of (weak) compositions of  $d$  into  $r+2$  parts (where “weak” means a possible zero part is allowed) by

$$\Lambda = \Lambda_{r,d} = \left\{ \lambda = (\lambda_0, \lambda_1, \dots, \lambda_{r+1}) \in \mathbb{N}^{r+2} \mid \sum_{i=0}^{r+1} \lambda_i = d \right\}. \tag{2.2.1}$$

For  $\lambda \in \Lambda$ , we shall denote by  $W_\lambda$  the parabolic (finite) subgroup of  $W$  generated by  $S \setminus \{s_{\lambda_0}, s_{\lambda_{0,1}}, \dots, s_{\lambda_{0,r}}\}$ , where  $\lambda_{0,i} = \lambda_0 + \lambda_1 + \dots + \lambda_i$  for  $0 \leq i \leq r$ ; note  $\lambda_{0,0} = \lambda_0$  and  $\lambda_{0,r} = d - \lambda_{r+1}$ . We define the integral intervals  $R_i^\lambda$  by

$$R_i^\lambda = \begin{cases} [-\lambda_0, \lambda_0] & \text{if } i = 0, \\ (\lambda_{0,i-1}, \lambda_{0,i}] & \text{if } i \in [1, r], \\ [d+1-\lambda_{r+1}, d+1+\lambda_{r+1}] & \text{if } i = r+1. \end{cases} \tag{2.2.2}$$

We further extend the definition of  $R_i^\lambda$  for all  $i \in \mathbb{Z}$  recursively by letting

$$R_{-i}^\lambda = \{-x \mid x \in R_i^\lambda\}, \quad R_{i+n}^\lambda = \{x + D \mid x \in R_i^\lambda\}. \tag{2.2.3}$$

Then the sets  $\{R_i^\lambda\}_{i \in \mathbb{Z}}$  partition the set  $\mathbb{Z}$ , that is,

$$R_i^\lambda \cap R_j^\lambda = \emptyset \quad \text{for } i \neq j, \quad \mathbb{Z} = \bigsqcup_{i \in \mathbb{Z}} R_i^\lambda.$$

Denote by  $\text{Stab}(X)$  the stabilizer subgroup of the action of  $W \equiv \text{Perm}^c(\mathbb{Z})$  on  $\mathbb{Z}$ , for any subset  $X \subset \mathbb{Z}$ .

**Lemma 2.2.** *For any  $\lambda \in \Lambda$ , we have  $W_\lambda = \bigcap_{i=0}^{r+1} \text{Stab}(R_i^\lambda)$ .*

*Proof.* By [BB05, Proposition 8.4.4] we have, for each  $0 \leq i \leq r$ ,

$$W_{S \setminus \{s_{\lambda_{0,i}}\}} = \text{Stab}([-\lambda_{0,i}, \lambda_{0,i}]) \cap \text{Stab}([\lambda_{0,i} + 1, D - \lambda_{0,i} - 1]).$$

The lemma follows by taking the intersection  $W_\lambda = \bigcap_{i=0}^r W_{S \setminus \{s_{\lambda_{0,i}}\}}$ .  $\square$



Let

$$\mathcal{D}_\lambda = \{g \in W \mid \ell(wg) = \ell(w) + \ell(g), \forall w \in W_\lambda\}. \quad (2.2.4)$$

Then  $\mathcal{D}_\lambda$  (respectively,  $\mathcal{D}_\lambda^{-1}$ ) is the set of minimal length right (respectively, left) coset representatives of  $W_\lambda$  in  $W$ . Denote by

$$\mathcal{D}_{\lambda\mu} = \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1} \quad (2.2.5)$$

the set of minimal length double coset representatives for  $W_\lambda \backslash W / W_\mu$ .

**Lemma 2.3.** *For any  $g \in W$  and  $\lambda \in \Lambda$ , the following are equivalent:*

- (a)  $g \in \mathcal{D}_\lambda$ ;
- (b)  $g^{-1}$  is order-preserving on  $R_i^\lambda$ , for all  $i \in [0..r+1]$ ;
- (c)  $g^{-1}$  is order-preserving on  $R_i^\lambda$ , for all  $i \in \mathbb{Z}$ .

*Proof.* By the argument following [BB05, Proposition 8.4.4],  $\mathcal{D}_\lambda$  in (2.2.4) can be written as

$$\mathcal{D}_\lambda = \left\{ g \in W \mid \begin{array}{l} g^{-1}(0) < \dots < g^{-1}(\lambda_0), \\ g^{-1}(1 + \lambda_{0,i}) < \dots < g^{-1}(\lambda_{0,i+1}), \forall i \in [1..r-1] \\ g^{-1}(1 + \lambda_{0,r}) < \dots < g^{-1}(d+1) \end{array} \right\}.$$

Note that  $g(-i) = -g(i)$  and  $g(0) = 0$ , so the condition “ $g^{-1}(0) < \dots < g^{-1}(\lambda_0)$ ” is equivalent to  $g^{-1}(-\lambda_0) < \dots < g^{-1}(0) = 0 < \dots < g^{-1}(\lambda_0)$ . Similarly, for  $g \in \mathcal{D}_\lambda$ , we have  $g^{-1}(d+1 - \lambda_{r+1}) < \dots < g^{-1}(d+1) = d+1 < \dots < g^{-1}(d+1 + \lambda_{r+1})$ . Hence (a) is equivalent to (b).

The equivalence of (b) and (c) follows from the periodicity condition (2.1.2).  $\square$

The following proposition is standard and can be found in [DDPW, Proposition 4.16, Lemma 4.17 and Theorem 4.18]; for Part (a) also see Proposition 2.9 below.

**Proposition 2.4.** *Let  $\lambda, \mu \in \Lambda$  and  $g \in \mathcal{D}_{\lambda\mu}$ .*

- (a) *There is a weak composition  $\delta = \delta(\lambda, g, \mu) \in \Lambda_{r',d}$  for some  $r'$  such that*

$$W_\delta = g^{-1}W_\lambda g \cap W_\mu.$$

- (b) *The map  $W_\lambda \times (\mathcal{D}_\delta \cap W_\mu) \rightarrow W_\lambda g W_\mu$  sending  $(x, y)$  to  $xgy$  is a bijection; moreover, we have  $\ell(xgy) = \ell(x) + \ell(g) + \ell(y)$ .*
- (c) *The map  $(\mathcal{D}_\delta \cap W_\mu) \times W_\delta \rightarrow W_\mu$  sending  $(x, y)$  to  $xy$  is a bijection; moreover, we have  $\ell(x) + \ell(y) = \ell(xy)$ .*

**2.3. Affine Schur algebra via Hecke.** Let  $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ . The Hecke algebra  $\mathbf{H} = \mathbf{H}(W)$  is an  $\mathcal{A}$ -algebra with a basis  $\{T_g \mid g \in W\}$  satisfying

$$\begin{aligned} T_w T_{w'} &= T_{ww'} && \text{if } \ell(ww') = \ell(w) + \ell(w'), \\ (T_s + 1)(T_s - v^2) &= 0, && \text{for } s \in S. \end{aligned}$$

For any finite subset  $X \subset W$  and for  $\lambda \in \Lambda$  (2.2.1), set

$$T_X = \sum_{w \in X} T_w \quad \text{and} \quad x_\lambda = T_{W_\lambda}. \quad (2.3.1)$$

For  $\lambda, \mu \in \Lambda$  and  $g \in \mathcal{D}_{\lambda\mu}$ , we consider a right  $\mathbf{H}$ -linear map  $\phi_{\lambda\mu}^g \in \text{Hom}_{\mathbf{H}}(x_\mu \mathbf{H}, \mathbf{H})$ , sending  $x_\mu$  to  $T_{W_\lambda g W_\mu}$ . Thanks to Proposition 2.4(b), we have  $T_{W_\lambda g W_\mu} = x_\lambda T_g T_{\mathcal{D}_\delta \cap W_\mu}$  for some  $\delta \in \Lambda_{r',d}$ , and hence we have constructed a right  $\mathbf{H}$ -linear map

$$\phi_{\lambda\mu}^g \in \text{Hom}_{\mathbf{H}}(x_\mu \mathbf{H}, x_\lambda \mathbf{H}), \quad x_\mu \mapsto T_{W_\lambda g W_\mu} = x_\lambda T_g T_{\mathcal{D}_\delta \cap W_\mu}. \quad (2.3.2)$$

The *affine Schur algebra*  $\mathbf{S}_{n,d}^c$  is defined as the following  $\mathcal{A}$ -algebra

$$\mathbf{S}_{n,d}^c = \text{End}_{\mathbf{H}}\left(\bigoplus_{\lambda \in \Lambda} x_\lambda \mathbf{H}\right) = \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_{\mathbf{H}}(x_\mu \mathbf{H}, x_\lambda \mathbf{H}). \quad (2.3.3)$$

Introduce the following subset of  $\Lambda \times W \times \Lambda$ :

$$\mathcal{D}_{n,d} = \bigsqcup_{\lambda, \mu \in \Lambda} \{\lambda\} \times \mathcal{D}_{\lambda, \mu} \times \{\mu\}. \quad (2.3.4)$$

A formal argument as in [Du92, Gr97] is applicable to our setting and gives us the following.

**Lemma 2.5.** *The set  $\{\phi_{\lambda\mu}^g \mid (\lambda, g, \mu) \in \mathcal{D}_{n,d}\}$  forms an  $\mathcal{A}$ -basis of  $\mathbf{S}_{n,d}^c$ .*

We denote

$$\Theta_n = \{A = (a_{ij}) \in \text{Mat}_{\mathbb{Z} \times \mathbb{Z}}(\mathbb{N}) \mid a_{ij} = a_{i+n, j+n}, \forall i, j \in \mathbb{Z}\}.$$

We further consider the following subset of  $\Theta_n$ :

$$\begin{aligned} \Xi_{n,d} = \left\{ A = (a_{ij}) \in \text{Mat}_{\mathbb{Z} \times \mathbb{Z}}(\mathbb{N}) \mid a_{-i, -j} = a_{ij} = a_{i+n, j+n}, \forall i, j \in \mathbb{Z}; \right. \\ \left. a_{00}, a_{r+1, r+1} \text{ are odd; } \sum_{1 \leq i \leq n} \sum_{j \in \mathbb{Z}} a_{ij} = D \right\}. \end{aligned} \quad (2.3.5)$$

For  $k, \ell \in \mathbb{Z}$ , we define a matrix  $E^{k\ell} \in \Theta_n$  by

$$E^{k\ell} = (E_{ij}^{k\ell})_{i, j \in \mathbb{Z}}, \quad E_{ij}^{k\ell} = \begin{cases} 1 & \text{if } (i, j) = (k + tn, \ell + tn) \text{ for some } t \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \quad (2.3.6)$$

For any  $T = (t_{ij}) \in \Theta_n$ , set

$$T_\theta = (t_{\theta, ij}), \quad t_{\theta, ij} = t_{ij} + t_{-i, -j}. \quad (2.3.7)$$

For  $T = (t_{ij}) \in \Theta_n$ , define its type A row sum vector  $\text{row}_a(T) = (\text{row}_a(T)_k)_{k \in \mathbb{Z}}$  and type A column sum vector  $\text{col}_a(T) = (\text{col}_a(T)_k)_{k \in \mathbb{Z}}$  by

$$\text{row}_a(T)_k = \sum_{j \in \mathbb{Z}} t_{kj} \quad \text{and} \quad \text{col}_a(T)_k = \sum_{i \in \mathbb{Z}} t_{ik}. \quad (2.3.8)$$

Let

$$\Xi_n = \bigcup_{d \in \mathbb{N}} \Xi_{n,d}. \quad (2.3.9)$$

For  $A = (a_{ij}) \in \Xi_n$ , we set

$$a'_{ij} = \begin{cases} \frac{1}{2}(a_{ii} - 1) & \text{if } i = j \in \mathbb{Z}(r+1), \\ a_{ij} & \text{otherwise.} \end{cases} \quad (2.3.10)$$

For any  $A \in \Xi_n$ , we define its type C row sum vector  $\text{row}_c(A) = (\text{row}_c(A)_0, \dots, \text{row}_c(A)_{r+1})$  and type C column sum vector  $\text{col}_c(A) = (\text{col}_c(A)_0, \dots, \text{col}_c(A)_{r+1})$  by

$$\begin{aligned} \text{row}_c(A)_k &= \begin{cases} a'_{00} + \sum_{j \geq 1} a_{0j} & \text{if } k = 0, \\ a'_{r+1,r+1} + \sum_{j \leq r} a_{r+1,j} & \text{if } k = r+1, \\ \text{row}_a(T)_k & \text{if } 1 \leq k \leq r. \end{cases} \\ \text{col}_c(A)_k &= \begin{cases} a'_{00} + \sum_{i \geq 1} a_{i0} & \text{if } k = 0, \\ a'_{r+1,r+1} + \sum_{i \leq r} a_{i,r+1} & \text{if } k = r+1, \\ \text{col}_a(A)_k & \text{if } 1 \leq k \leq r. \end{cases} \end{aligned} \quad (2.3.11)$$

Each  $A \in \Xi_n$  is uniquely determined by  $\{a_{ij} \mid (i, j) \in I^+\}$ , where

$$I^+ = (\{0\} \times \mathbb{N}) \sqcup ([1..r] \times \mathbb{Z}) \sqcup (\{r+1\} \times \mathbb{Z}_{\leq r+1}) \quad (2.3.12)$$

is the index set corresponding to the “first half-period”.

**2.4. Set-valued matrices.** We introduce a “higher-level” structure of  $\Xi_n$ , which will facilitate the proof of the multiplication formula in next section. Let  $\Xi_{n,d}^{\mathcal{P}}$  be the set of  $\mathbb{Z} \times \mathbb{Z}$  matrices  $\mathbb{A} = (\mathbb{A}_{ij})$  with entries being finite subsets of  $\mathbb{Z}$  which satisfy Conditions (P1)–(P5) below:

- (P1) For  $z \in \mathbb{Z}$ , there exists a unique  $(i, j)$  such that  $z \in \mathbb{A}_{ij}$ ;
- (P2)  $\mathbb{A}_{i+n,j+n} = \{x + D \mid x \in \mathbb{A}_{ij}\}$  for all  $i, j \in \mathbb{Z}$ ;
- (P3)  $\mathbb{A}_{-i,-j} = \{-x \mid x \in \mathbb{A}_{ij}\}$  for all  $i, j \in \mathbb{Z}$ ;
- (P4)  $0 \in \mathbb{A}_{00}$  and  $d+1 \in \mathbb{A}_{r+1,r+1}$ ;
- (P5)  $\sum_{1 \leq i \leq n} \sum_{j \in \mathbb{Z}} |\mathbb{A}_{ij}| = D$ .

Set

$$\Xi_n^{\mathcal{P}} = \bigsqcup_{d \in \mathbb{N}} \Xi_{n,d}^{\mathcal{P}}.$$

Again, each  $\mathbb{A} \in \Xi_n^{\mathcal{P}}$  is uniquely determined by  $\{\mathbb{A}_{ij} \mid (i, j) \in I^+\}$ .

Recalling  $I^+$  in (2.3.12), we denote

$$I_a^+ = I^+ \setminus \{(0, 0), (r+1, r+1)\}. \quad (2.4.1)$$

We define a map

$$\mathcal{P} : \Xi_{n,d} \longrightarrow \Xi_{n,d}^{\mathcal{P}}, \quad A \mapsto A^{\mathcal{P}} \quad (2.4.2)$$

as follows. For each  $A = (a_{ij}) \in \Xi_{n,d}$ , a matrix  $A^{\mathcal{P}} \in \Xi_{n,d}^{\mathcal{P}}$  is determined by (a1)–(a2) below via “row-reading” (thanks to Conditions (P2)–(P3) for  $\Xi_{n,d}^{\mathcal{P}}$ ):

- (a1) Set  $(A^{\mathcal{P}})_{00} = [-a'_{00} \dots a'_{00}]$ ,  $(A^{\mathcal{P}})_{r+1,r+1} = [d+1 - a'_{r+1,r+1} \dots d+1 + a'_{r+1,r+1}]$ .
- (a2) For  $(i, j) \in I_a^+$ , set

$$(A^{\mathcal{P}})_{ij} = \left( \sum_{l=0}^{i-1} \text{row}_c(A)_l + \sum_{k < j} a_{ik} \dots \sum_{l=0}^{i-1} \text{row}_c(A)_l + \sum_{k \leq j} a_{ik} \right).$$

By definition we have a map

$$|\cdot| : \Xi_{n,d}^{\mathcal{P}} \longrightarrow \Xi_{n,d}, \quad |\mathbb{A}| = (|\mathbb{A}_{ij}|)_{i,j \in \mathbb{Z}}. \quad (2.4.3)$$

Moreover,  $|A^{\mathcal{P}}| = A$ .

We further define a map

$$g^{\text{std}} : \Xi_{n,d}^{\mathcal{P}} \longrightarrow W, \quad \mathbb{A} \mapsto g_{\mathbb{A}}^{\text{std}} \quad (2.4.4)$$

as follows. On  $I^+$ , let  $<_{\text{lex}}$  be the lexicographical order such that  $(i, j) <_{\text{lex}} (x, y)$  if and only if  $i < x$  or  $(i = x \text{ and } j < y)$ . Let  $\mathbb{A} = (\mathbb{A}_{ij}) \in \Xi_{n,d}^{\mathcal{P}}$ , and set  $A = (a_{ij}) = |\mathbb{A}|$ . The Weyl group element  $g_{\mathbb{A}}^{\text{std}} \in W$  is determined by (g1)–(g2) below via “column-reading” (thanks to the periodicity condition (2.1.2)):

(g1) For  $(i, j) \in I^+$ , we set

$$I^{(j,i)} = \begin{cases} [-a'_{ii}..a'_{ii}], & \text{if } (j, i) = (0, 0) \text{ or } (r+1, r+1), \\ [1..a_{ji}], & \text{otherwise.} \end{cases}$$

Then set  $\mathbb{A}_{j,i} = \{a_l^{(j,i)} \mid l \in I^{(j,i)}\}$  such that  $a_l^{(j,i)} < a_{l+1}^{(j,i)}$  for admissible  $l$ .

(g2) Let  $1 \leq k \leq d$ . If  $k \leq a'_{00}$ , set  $g_{\mathbb{A}}^{\text{std}}(k) = a_m^{(0,0)}$ ; if  $k > a'_{00}$ , then find the unique  $(i, j) \in I^+ \setminus \{(0, 0)\}$  and  $m \in [1..a'_{ji}]$  such that

$$k = a'_{00} + \sum_{(x,y) \in I^+ \setminus \{(0,0)\}, (x,y) <_{\text{lex}} (i,j)} a_{yx} + m,$$

and then set  $g_{\mathbb{A}}^{\text{std}}(k) = a_m^{(j,i)}$ .

**Example 2.6.** Let  $d = 3, D = 8, r = 0, n = 2$ , and let

$$A = \left[ \begin{array}{ccc|ccc} \ddots & & & & & \\ & 2 & 0 & 1 & 0 & 2 \\ \hline & & & 3 & & \\ \hline & & & 2 & 0 & 1 & 0 & 2 \\ & & & & & & 3 & \\ & & & & & & & \ddots \end{array} \right] = 3E^{00} + 2E_{\theta}^{1,-1} + E^{11}.$$

Here the dashed stripes indicate the 0th column/row. We have  $\text{row}_c(A) = \text{col}_c(A) = (1, 2)$ , and

$$A^{\mathcal{P}} = \left[ \begin{array}{ccc|ccc} \ddots & & & & & \\ \{-6, -5\} & \emptyset & \{-4\} & \emptyset & \{-3, -2\} & \\ \hline & & & [-1..1] & & \\ \hline & & \{2, 3\} & \emptyset & \{4\} & \emptyset & \{5, 6\} \\ & & & & & [7..9] & \\ & & & & & & \ddots \end{array} \right].$$

On the other hand, for  $\mathbb{A} = A^{\mathcal{P}}$ , we have  $\mathbb{A}_{0,0} = [-1..1]$ ,  $\mathbb{A}_{-1,1} = \{-3, -2\}$ ,  $\mathbb{A}_{1,1} = \{4\}$ . Hence for  $(i, j) \in I^+$ ,  $a_m^{(j,i)}$  are defined by

$$a_{-1}^{(0,0)} = -1 < a_0^{(0,0)} = 0 < a_1^{(0,0)} = 1, \quad a_1^{(-1,1)} = -3 < a_2^{(-1,1)} = -2, \quad a_0^{(1,1)} = 4.$$

Therefore,  $g_{A^{\mathcal{P}}}^{\text{std}} = [1, -3, -2]_c$ ; see (2.1.3).

**2.5. A bijection  $\kappa$ .** For  $\lambda, \mu \in \Lambda$ , we denote

$$\Xi_{n,d}(\lambda, \mu) = \{A \in \Xi_{n,d} \mid \text{row}_c(A) = \lambda, \text{col}_c(A) = \mu\}.$$

Then we have  $\Xi_{n,d} = \bigsqcup_{\lambda, \mu \in \Lambda} \Xi_{n,d}(\lambda, \mu)$ . We define a map

$$\kappa_{\lambda\mu} : W \longrightarrow \Xi_{n,d}(\lambda, \mu), \quad g \mapsto (|R_i^\lambda \cap gR_j^\mu|). \quad (2.5.1)$$

Recalling  $\Lambda$  from (2.2.1), we can assemble the maps  $\kappa_{\lambda\mu}$  above for various  $\lambda, \mu$  to a map

$$\kappa : \Lambda \times W \times \Lambda \longrightarrow \Xi_{n,d}, \quad \kappa(\lambda, g, \mu) = \kappa_{\lambda,\mu}(g).$$

By restriction to  $\mathcal{D}_{n,d}$  (2.3.4), we obtain a map

$$\kappa : \mathcal{D}_{n,d} \longrightarrow \Xi_{n,d}, \quad \kappa(\lambda, g, \mu) = \kappa_{\lambda,\mu}(g). \quad (2.5.2)$$

**Lemma 2.7.** *The map  $\kappa : \mathcal{D}_{n,d} \longrightarrow \Xi_{n,d}$  with  $\kappa(\lambda, g, \mu) = (|R_i^\lambda \cap gR_j^\mu|)$  is a bijection. Moreover, if  $(\lambda, g, \mu) = \kappa^{-1}(A)$  for  $A \in \Xi_{n,d}$ , then  $\lambda = \text{row}_c(A)$ ,  $\mu = \text{col}_c(A)$ , and  $g = g_{A^P}^{\text{std}}$ .*

*Proof.* We shall show equivalently that for fixed  $\lambda, \mu \in \Lambda$ , the restriction of  $\kappa_{\lambda\mu}$  to  $\mathcal{D}_{\lambda\mu}$ ,  $\kappa_{\lambda\mu}|_{\mathcal{D}_{\lambda\mu}} : \mathcal{D}_{\lambda\mu} \longrightarrow \Xi_{n,d}(\lambda, \mu)$ , is a bijection.

Let  $\Xi_d^{\mathcal{P}}(\lambda, \mu) = \{\mathbb{A} \in \Xi_{n,d}^{\mathcal{P}} \mid \text{row}_c(|\mathbb{A}|) = \lambda, \text{col}_c(|\mathbb{A}|) = \mu\}$ . Consider the map

$$\kappa_{\lambda\mu}^{\mathcal{P}} : W \longrightarrow \Xi_d^{\mathcal{P}}(\lambda, \mu), \quad g \mapsto (R_i^\lambda \cap gR_j^\mu),$$

so we have  $\kappa_{\lambda\mu} = |\cdot| \circ \kappa_{\lambda\mu}^{\mathcal{P}}$ . The map  $\kappa_{\lambda\mu}^{\mathcal{P}}$  is a surjection since  $\kappa_{\lambda\mu}^{\mathcal{P}}(g_{\mathbb{A}}^{\text{std}}) = \mathbb{A}$  for  $\mathbb{A} \in \Xi_d^{\mathcal{P}}(\lambda, \mu)$ . Furthermore, given  $g \in (\kappa_{\lambda\mu}^{\mathcal{P}})^{-1}(\mathbb{A})$ , by Lemma 2.3 we have  $g = g_{\mathbb{A}}^{\text{std}}$  if and only if  $g \in \mathcal{D}_{\lambda\mu}$ . Thus the restriction  $\kappa_{\lambda\mu}^{\mathcal{P}}|_{\mathcal{D}_{\lambda\mu}}$  is a bijection. On the other hand, for each  $A \in \Xi_{n,d}(\lambda, \mu)$  we have  $|A^{\mathcal{P}}| = A$ , and hence  $\kappa_{\lambda\mu} = (|\cdot| \circ \kappa_{\lambda\mu}^{\mathcal{P}})_{\lambda\mu}$  is a surjection.

Moreover, given  $\mathbb{A} \in \Xi_d^{\mathcal{P}}(\lambda, \mu)$  with  $|\mathbb{A}| = A$ , applying Lemma 2.3 again we have  $\mathbb{A} = A^{\mathcal{P}}$  if and only if  $g_{\mathbb{A}}^{\text{std}} \in \mathcal{D}_{\lambda\mu}$ . Therefore  $(|\cdot| \circ \kappa_{\lambda\mu}^{\mathcal{P}})|_{\mathcal{D}_{\lambda\mu}} = \kappa_{\lambda\mu}|_{\mathcal{D}_{\lambda\mu}}$  is a bijection.

The second statement of the lemma can be read off from the above argument.  $\square$

For each  $A = \kappa(\lambda, g, \mu) \in \Xi_{n,d}$ , we use the bijection  $\kappa$  in Lemma 2.7 to introduce new notation

$$e_A = \phi_{\lambda\mu}^g. \quad (2.5.3)$$

Hence Lemma 2.5 can be rephrased that  $\{e_A \mid A \in \Xi_{n,d}\}$  forms an  $\mathcal{A}$ -basis of  $\mathbf{S}_{n,d}^c$ .

Let  $A \in \Xi_{n,d}$ . We choose  $k_j \geq 0$  for  $0 \leq j \leq r+1$  such that  $a_{ij} = 0$  unless  $|i-j| \leq k_j$ . We define a weak composition

$$\delta(A) \in \Lambda_{\mathbf{r},d}, \quad (2.5.4)$$

with  $\mathbf{r} = k_0 + \sum_{j=1}^r (2k_j + 1) + k_{r+1}$  as follows. The composition  $\delta(A)$  starts with the  $(k_0 + 1)$  entries in 0th column,  $a'_{00}, a_{10}, a_{20}, \dots, a_{k_0,0}$ , followed by the  $(2k_j + 1)$  entries  $a_{ij}$  in  $j$ th column when  $i$  runs up the interval  $[j-k_j, j+k_j]$  for  $j = 1, \dots, r$ , and finally followed by the  $(k_{r+1} + 1)$  entries in the  $(r+1)$ st column,  $a_{r+1-k_{r+1}, r+1}, \dots, a_{r, r+1}, a'_{r+1, r+1}$ .

**Remark 2.8.** We can remove any zeroes that are not in the first or the last place in a weak composition  $\lambda \in \Lambda_{\mathbf{r},d}$  without changing the parabolic subgroup  $W_\lambda$ , e.g.,  $W_{(2,0,2)} = W_{(2,2)}$ . (However, removing zeroes in the first or the last place in a weak composition changes the corresponding parabolic subgroup.) Therefore, while the composition  $\delta(A)$  in (2.5.4) depends on  $A$  as well as on the choices of  $k_j$ , the parabolic subgroup  $W_{\delta(A)}$  only depends on  $A$ .

We can now make the construction in Proposition 2.4(a) explicit.

**Proposition 2.9.** *Let  $A = \kappa(\lambda, g, \mu)$  for  $\lambda, \mu \in \Lambda, g \in \mathcal{D}_{\lambda\mu}$ . Then  $W_{\delta(A)} = g^{-1}W_{\lambda}g \cap W_{\mu}$ .*

*Proof.* By Lemma 2.2, we have

$$\begin{aligned} g^{-1}W_{\lambda}g \cap W_{\mu} &= \left( \bigcap_{i=0}^{r+1} \text{Stab}(g^{-1}R_i^{\lambda}) \right) \cap \left( \bigcap_{j=0}^{r+1} \text{Stab}(R_j^{\mu}) \right) \\ &= \bigcap_{(i,j) \in I^+} \text{Stab}(g^{-1}R_i^{\lambda} \cap R_j^{\mu}) = \bigcap_{(i,j) \in I^+} \text{Stab}(g^{-1}(A^P)_{ij}) = W_{\delta(A)}. \end{aligned}$$

The proposition is proved.  $\square$

**2.6. Computation in affine Schur algebra  $S_{n,d}^c$ .** We denote the (type A) quantum  $v$ -number and quantum  $v$ -factorial by, for  $m \in \mathbb{Z}, n \in \mathbb{N}$ ,

$$[m] = \frac{v^{2m} - 1}{v^2 - 1}, \quad [n]^! = [n][n-1] \cdots [1], \quad (2.6.1)$$

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{[m][m-1] \cdots [m-n+1]}{[n]^!}. \quad (2.6.2)$$

(It is understood that  $[0]^! = 1$ .) For  $T = (t_{ij}) \in \Theta_n$ , we define

$$[T]^! = \prod_{i=1}^n \prod_{j \in \mathbb{Z}} [t_{ij}]^!. \quad (2.6.3)$$

For  $A = (a_{ij}) \in \Xi_n$ , define (see (2.3.10) for  $a'_{ij}$ )

$$[A]_{\mathfrak{c}}^! = [a'_{00}]_{\mathfrak{c}}^! [a'_{r+1,r+1}]_{\mathfrak{c}}^! \cdot \prod_{(i,j) \in I_{\mathfrak{a}}^+} [a_{ij}]^!, \quad (2.6.4)$$

where  $[m]_{\mathfrak{c}}^! = \prod_{k=1}^m [2k]$ . In particular, we have  $[a'_{ii}]_{\mathfrak{c}}^! = [2] \cdot [4] \cdots [a_{ii} - 1]$  for  $i = 0, r+1$ .

**Lemma 2.10.** *For any  $A \in \Xi_n$ , we have  $[A]_{\mathfrak{c}}^! = \sum_{w \in W_{\delta(A)}} v^{2\ell(w)}$ .*

*Proof.* Denote the Weyl group of type  $A_{m-1}$  (respectively,  $C_m$ ) by  $S_m$  (respectively,  $W_{C_m}$ ). It is well known that the Poincare polynomial for  $S_m$  and  $W_{C_m}$  are, respectively,

$$\sum_{w \in S_m} v^{2\ell(w)} = \prod_{k=1}^m [k] = [m]^! \quad \text{and} \quad \sum_{w \in W_{C_m}} v^{2\ell(w)} = \prod_{k=1}^m [2k] = [m]_{\mathfrak{c}}^!.$$

Since  $W_{\delta(A)} \simeq W_{C_{\delta_0}} \times S_{\delta_1} \times S_{\delta_2} \times \cdots \times S_{\delta_{\mathfrak{r}}} \times W_{C_{\delta_{\mathfrak{r}+1}}}$ , we obtain

$$\sum_{w \in W_{\delta(A)}} v^{2\ell(w)} = \prod_{i \in \{0, \mathfrak{r}+1\}} \left( \sum_{w \in W_{C_{\delta_i}}} v^{2\ell(w)} \right) \prod_{i=1}^{\mathfrak{r}} \left( \sum_{w \in S_{\delta_i}} v^{2\ell(w)} \right) = [A]_{\mathfrak{c}}^!.$$

The lemma is proved.  $\square$

**Lemma 2.11.** *Let  $A = \kappa(\lambda, g, \mu)$  for  $\lambda, \mu \in \Lambda, g \in \mathcal{D}_{\lambda\mu}$ . Then  $x_{\lambda}T_g x_{\mu} = [A]_{\mathfrak{c}}^! e_A(x_{\mu})$ .*

*Proof.* Let  $\delta = \delta(A)$ . By Proposition 2.4(c), we have

$$x_\mu = \sum_{x \in W_\mu} T_x = \sum_{\substack{w \in \mathcal{D}_\delta \cap W_\mu \\ y \in W_\delta}} T_{wy} = \sum_{w \in \mathcal{D}_\delta \cap W_\mu} T_w \sum_{y \in W_\delta} T_y = T_{\mathcal{D}_\delta \cap W_\mu} x_\delta.$$

Note that  $x_\mu T_w = v^{2\ell(w)} x_\mu$  for any  $w \in W_\mu$ , and thus it follows by Lemma 2.10 and  $W_\delta \subset W_\mu$  that  $x_\mu x_\delta = \sum_{w \in W_\delta} v^{2\ell(w)} x_\mu = [A]_c^! x_\mu$ . Therefore by (2.3.2) and (2.5.3) we have

$$x_\lambda T_g x_\mu = x_\lambda T_g T_{\mathcal{D}_\delta \cap W_\mu} x_\delta = e_A(x_\mu) x_\delta = e_A(x_\mu x_\delta) = [A]_c^! e_A(x_\mu).$$

The lemma is proved.  $\square$

**Lemma 2.12.** *Let  $B = \kappa(\lambda, g_1, \mu)$  and  $A = \kappa(\mu, g_2, \nu)$ , where  $\lambda, \mu, \nu \in \Lambda$ ,  $g_1 \in \mathcal{D}_{\lambda\mu}$ , and  $g_2 \in \mathcal{D}_{\mu\nu}$ . Let  $\delta = \delta(B)$ ; see (2.5.4). Then we have*

$$e_B e_A(x_\nu) = \frac{1}{[A]_c^!} x_\lambda T_{g_1} T_{(\mathcal{D}_\delta \cap W_\mu)g_2} x_\nu.$$

*Proof.* It follows by Lemma 2.11, (2.3.2) and (2.5.3) that

$$e_B e_A(x_\nu) = \frac{1}{[A]_c^!} e_B(x_\mu T_{g_2} x_\nu) = \frac{1}{[A]_c^!} e_B(x_\mu) T_{g_2} x_\nu = \frac{1}{[A]_c^!} x_\lambda T_{g_1} T_{\mathcal{D}_\delta \cap W_\mu} T_{g_2} x_\nu.$$

Since  $g_2 \in \mathcal{D}_\mu^{-1}$ , we have  $T_w T_{g_2} = T_{wg_2}$  for all  $w \in \mathcal{D}_\delta \cap W_\mu$ . Therefore  $T_{\mathcal{D}_\delta \cap W_\mu} T_{g_2} = T_{(\mathcal{D}_\delta \cap W_\mu)g_2}$  and we are done.  $\square$

For  $w \in W_\mu$ , although  $T_{g_1} T_w = T_{g_1 w}$  and  $T_w T_{g_2} = T_{wg_2}$ , it is not true that  $T_{g_1} T_w T_{g_2} = T_{g_1 w g_2}$  in general. Therefore we need to write out  $T_{g_1} T_{wg_2}$  in order to have a useful multiplication formula. For  $w \in \mathcal{D}_\delta \cap W_\mu$ , we write

$$T_{g_1} T_{wg_2} = \sum_{\sigma \in \Delta(w)} c^{(w, \sigma)} T_{g_1 \sigma w g_2}, \quad \text{for } c^{(w, \sigma)} \in \mathbb{Z}[v^2] \text{ and finite subset } \Delta(w) \subset W. \quad (2.6.5)$$

For  $\sigma \in \Delta(w)$ , we have

$$T_{g_1 \sigma w g_2} = T_{w_\lambda^{(\sigma)}} T_{y^{(w, \sigma)}} T_{w_\nu^{(\sigma)}} \quad (2.6.6)$$

if we write

$$g_1 \sigma w g_2 = w_\lambda^{(\sigma)} y^{(w, \sigma)} w_\nu^{(\sigma)}, \quad \text{for some } y^{(w, \sigma)} \in \mathcal{D}_{\lambda\nu}, w_\lambda^{(\sigma)} \in W_\lambda, w_\nu^{(\sigma)} \in W_\nu. \quad (2.6.7)$$

We further denote

$$A^{(w, \sigma)} = (a_{ij}^{(w, \sigma)}) = \kappa(\lambda, y^{(w, \sigma)}, \nu). \quad (2.6.8)$$

**Proposition 2.13.** *Let  $\delta = \delta(B)$ ; see (2.5.4). Let  $c^{(w, \sigma)}, \Delta(w)$  and  $A^{(w, \sigma)}$  be defined as in (2.6.5) and (2.6.8). Then we have*

$$e_B e_A = \sum_{\substack{w \in \mathcal{D}_\delta \cap W_\mu \\ \sigma \in \Delta(w)}} c^{(w, \sigma)} v^{2\ell(g_1 \sigma w g_2) - 2\ell(y^{(w, \sigma)})} \frac{[A^{(w, \sigma)}]_c^!}{[A]_c^!} e_{A^{(w, \sigma)}}. \quad (2.6.9)$$

*Proof.* Combining Lemma 2.12 and (2.6.5), we have

$$e_B e_A(x_\nu) = \frac{1}{[A]_c!} \sum_{\substack{w \in \mathcal{D}_\delta \cap W_\mu \\ \sigma \in \Delta(w)}} c^{(w, \sigma)} x_\lambda T_{g_1 \sigma w g_2} x_\nu.$$

For  $\sigma \in \Delta(w)$ , by (2.6.6) we have  $x_\lambda T_{g_1 \sigma w g_2} x_\nu = v^{2\ell(w_\lambda^{(\sigma)}) + 2\ell(w_\nu^{(\sigma)})} x_\lambda T_{y(w, \sigma)} x_\nu$ . The proposition now follows by applying Lemma 2.11.  $\square$

It is unrealistic to obtain an explicit description for  $c^{(w, \sigma)}$  in (2.6.5) for general  $g_1$  since it amounts to obtaining explicitly all the structure constants for affine Hecke algebras. Later on we shall treat the special case when  $B$  is tridiagonal, when the structure constants will be computed explicitly.

**2.7. Isomorphism  $\mathbf{S}_{n,d}^{\text{c,geo}} \cong \mathbf{S}_{n,d}^{\text{c}}$ .** In this subsection we show that the Schur algebra  $\mathbf{S}_{n,d}^{\text{c}}$  defined in §2.3 can be identified with the affine Schur algebra in [FLLLW] which was defined as a convolution algebra in a geometric setting. We shall add superscript “geo” (a shorthand for “geometric”) to the notations used *loc. cit.*

Let  $F = \mathbb{F}_{q^2}((\epsilon))$  be the field of formal Laurent series over a finite field  $\mathbb{F}_{q^2}$  of  $q^2$  elements, and let  $\text{Sp}_F(2d)$  be the symplectic group with coefficients in  $F$ . Set  $\mathbf{S}_{n,d}^{\text{c,geo}}$  to be the Schur algebra in [FLLLW, §4.2] (denoted by  $\mathbf{S}_{n,d;A}$  therein) which specializes at  $v = q$  as

$$\mathbf{S}_{n,d}^{\text{c,geo}}|_{v=q} = \mathcal{A}_{\text{Sp}_F(2d)}(\mathcal{X}_{n,d}^{\text{c}} \times \mathcal{X}_{n,d}^{\text{c}}), \quad (2.7.1)$$

the convolution algebra on the set  $\mathcal{X}_{n,d}^{\text{c}}$  of  $n$ -step periodic lattices of affine type  $C$  in an  $F$ -vector space  $V$  of dimension  $2d$ . Recall that  $\Xi_{n,d}$  parameterizes  $\text{Sp}_F(2d)$ -orbits of  $\mathcal{X}_{n,d}^{\text{c}} \times \mathcal{X}_{n,d}^{\text{c}}$ . For any  $A \in \Xi_{n,d}$ , denote by  $e_A^{\text{geo}}$  the characteristic function on the orbit  $\mathcal{O}_A$ . The Hecke algebra  $\mathbf{H}$  which specializes at  $v = q$  as a convolution algebra

$$\mathbf{H}|_q = \mathcal{A}_{\text{Sp}_F(2d)}(\mathcal{Y}_d^{\text{c}} \times \mathcal{Y}_d^{\text{c}}), \quad (2.7.2)$$

where  $\mathcal{Y}_d^{\text{c}}$  is the set of ‘complete’ periodic lattices of affine type  $C$  in  $V$ .

**Proposition 2.14.** *There is an algebra isomorphism  $\psi : \mathbf{S}_{n,d}^{\text{c,geo}} \xrightarrow{\cong} \mathbf{S}_{n,d}^{\text{c}}$ , which sends  $e_A^{\text{geo}}$  to  $e_A$  for each  $A \in \Xi_{n,d}$ .*

*Proof.* We only need to prove the statement when  $v$  is specialized to  $q$  for various prime powers  $q$ . Let  $\psi : \mathbf{S}_{n,d}^{\text{c,geo}} \rightarrow \mathbf{S}_{n,d}^{\text{c}}$  be the  $\mathcal{A}$ -linear map sending  $e_A^{\text{geo}}$  to  $e_A$  for all  $A \in \Xi_{n,d}$ . Since  $\Xi_{n,d}$  parameterizes the basis of both algebras,  $\psi$  is an isomorphism.

It remains to show that  $\psi$  is an algebra homomorphism. Fix  $A, B, C \in \Xi_d$ , and let  $\lambda, \mu, \nu \in \Lambda$  and  $g_1, g_2, g_3 \in W$  be such that

$$A = \kappa(\lambda, g_1, \mu), \quad B = \kappa(\mu, g_2, \nu), \quad C = \kappa(\lambda, g_3, \nu).$$

Set  $g_{AB}^C(v) \in \mathcal{A}$  to be such that

$$g_{AB}^C(q) = \left| \left\{ \tilde{L} \in \mathcal{X}_{n,d}^{\text{c}} \mid (L, \tilde{L}) \in \mathcal{O}_A, (\tilde{L}, L') \in \mathcal{O}_B, (L, L') \in \mathcal{O}_C \right\} \right|, \quad (2.7.3)$$

for some fixed  $L, L' \in \mathcal{X}_{n,d}^{\text{c}}$ . Here  $\mathcal{O}_A$  is the  $\text{Sp}_F(2d)$ -orbit in  $\mathcal{X}_{n,d}^{\text{c}} \times \mathcal{X}_{n,d}^{\text{c}}$  indexed by  $A$ . Therefore, by definition we have

$$e_A^{\text{geo}} e_B^{\text{geo}} = \sum_C g_{AB}^C(v) e_C^{\text{geo}}. \quad (2.7.4)$$



For  $x, y, z \in W$ , set  $B_{xy}^z(v) \in \mathcal{A}$  to be such that

$$T_x T_y = \sum_z B_{xy}^z(v) T_z. \quad (2.7.5)$$

For  $g \in W$ , let  $\mathcal{O}_g$  be the orbit  $\mathcal{O}_{\kappa(\omega, g, \omega)}$  where  $\omega = (0, 1, 1, \dots, 1, 0) \in \Lambda_{d,d}$ . It is well known that

$$B_{xy}^z(q) = \left| \left\{ \tilde{L} \in \mathcal{Y}_d^c \mid (L, \tilde{L}) \in \mathcal{O}_x, (\tilde{L}, L') \in \mathcal{O}_y, (L, L') \in \mathcal{O}_z \right\} \right|,$$

for some fixed  $L, L' \in \mathcal{Y}_d^c$ . Then (as in [Du92, Proposition 3.4]) for any  $z \in W_{\lambda} g_3 W_{\nu}$  we have

$$g_{AB}^C(q) = \pi_{\mu}(q)^{-1} \sum_{\substack{x \in W_{\lambda} g_1 W_{\mu} \\ y \in W_{\mu} g_2 W_{\nu}}} B_{xy}^z(q), \quad (2.7.6)$$

where  $\pi_{\mu}(v) = \sum_{x \in W_{\mu}} v^{2\ell(x)}$ .

On the other hand, it is well known that (in particular it follows as a special case of Lemma 2.11 and its proof with  $A = \kappa(\mu, 1, \mu)$  and  $W_{\delta} = W_{\mu}$ )

$$x_{\mu}^2 = \pi_{\mu}(v) x_{\mu}. \quad (2.7.7)$$

Therefore by (2.3.2), (2.5.3) and (2.7.7) we have

$$\begin{aligned} e_A e_B(x_{\nu}) &= e_A(x_{\mu} T_{g_2} T_{\mathcal{D}_{\delta(B)} \cap W_{\nu}}) = e_A(x_{\mu}) T_{g_2} T_{\mathcal{D}_{\delta(B)} \cap W_{\nu}} \\ &= \pi_{\mu}(v)^{-1} e_A(x_{\mu}) x_{\mu} T_{g_2} T_{\mathcal{D}_{\delta(B)} \cap W_{\nu}} = \pi_{\mu}(v)^{-1} T_{W_{\lambda} g_1 W_{\mu}} T_{W_{\mu} g_2 W_{\nu}}, \end{aligned}$$

which, by (2.7.5) and (2.7.6), can be rewritten as

$$e_A e_B(x_{\nu}) = \pi_{\mu}(v)^{-1} \sum_{z \in W_{\lambda} g_3 W_{\nu}} \sum_{\substack{x \in W_{\lambda} g_1 W_{\mu} \\ y \in W_{\mu} g_2 W_{\nu}}} B_{xy}^z(v) T_z = \sum_{z \in W_{\lambda} g_3 W_{\nu}} g_{AB}^C(v) T_z.$$

This implies that  $e_A e_B = \sum_C g_{AB}^C(v) e_C$ , and hence  $\psi$  is an algebra homomorphism.  $\square$

### 3. MULTIPLICATION FORMULA FOR AFFINE HECKE ALGEBRA

In this section, we establish a multiplication formula in affine Hecke algebra  $\mathbf{H}$  with an element of the form  $T_g$ , where  $g \in \mathcal{D}_{\lambda\mu}$  for some  $\lambda, \mu \in \Lambda$  and  $\kappa(\lambda, g, \mu)$  is tridiagonal. This formula is used to obtain a corresponding multiplication formula for affine Schur algebra  $\mathbf{S}_{n,d}^c$  in Section 4.

**3.1. Minimal length representatives.** Fix  $\lambda, \mu \in \Lambda$ ,  $g_1 \in \mathcal{D}_{\lambda\mu}$ . Let  $B = \kappa(\lambda, g_1, \mu)$ . We assume that  $B = (b_{ij}) = \kappa(\lambda, g_1, \mu)$  is tridiagonal, i.e.,  $b_{ij} = 0$  unless  $|i - j| \leq 1$ . We choose (2.5.4) to be

$$\delta = \delta(B) = (b'_{00}, b_{10}; b_{01}, b_{11}, b_{21}; \dots, b_{r-1,r}, b_{r,r}, b_{r+1,r}; b_{r,r+1}, b'_{r+1,r+1}) \in \Lambda_{3r+2,d}. \quad (3.1.1)$$

Recalling the convention of indexing in (2.2.1), we understand that  $\delta$  has  $3r + 4$  components indexed by  $0, 1, \dots, 3r + 2, 3r + 3$ .

**Lemma 3.1.** *We have, for  $i \in \mathbb{Z}$ ,*

$$R_i^{\mu} = R_{3i-1}^{\delta} \cup R_{3i}^{\delta} \cup R_{3i+1}^{\delta}, \quad g_1^{-1} R_i^{\lambda} = R_{3i-2}^{\delta} \cup R_{3i}^{\delta} \cup R_{3i+2}^{\delta}.$$

*Proof.* Follows by the construction (2.2.2)–(2.2.3) and the definition of  $\delta$  in (3.1.1).  $\square$

Again, let  $B = \kappa(\lambda, g_1, \mu)$  be tridiagonal. By Lemma 2.7 we have  $g_1 = g_{B^{\text{std}}}^{\text{std}}$ . Unraveling the definition (2.4.4) for  $g_{B^{\text{std}}}^{\text{std}}$ , we can write  $g_1 = \prod_{i=1}^{r+1} g_1^{(i)}$ , where  $g_1^{(i)} \in W$  is specified by the following recipe:

$$g_1^{(i)}(x) = \begin{cases} x + b_{i-1,i} & \text{if } x \in R_{3i-2}^\delta \subset R_{i-1}^\mu, \\ x - b_{i,i-1} & \text{if } x \in R_{3i-1}^\delta \subset R_i^\mu, \\ x & \text{if } x \in [1..d] \setminus (R_{3i-2}^\delta \cup R_{3i-1}^\delta). \end{cases} \quad (3.1.2)$$

**Lemma 3.2.** *Fix  $1 \leq i \leq r+1$ . Write  $R_{3i-2}^\delta = [m+1..m+\alpha]$  and  $R_{3i-1}^\delta = [m+\alpha+1..m+\alpha+\beta]$  for some  $m, \alpha, \beta \in \mathbb{N}$ . Then  $g_1^{(i)}$  has a reduced expression*

$$g_1^{(i)} = (s_{m+\beta} \cdots s_{m+2} s_{m+1}) (s_{m+\beta+1} \cdots s_{m+2}) \cdots (s_{m+\beta+\alpha-1} \cdots s_{m+\alpha}). \quad (3.1.3)$$

*Proof.* For  $1 \leq t \leq \alpha$ , the product  $s_{m+\beta+t-1} \cdots s_{m+t}$  is the cyclic permutation on  $[m+t..m+t+\beta]$ , which sends  $m+t \mapsto m+t+1 \mapsto \dots \mapsto m+\beta+t \mapsto m+t$ . The lemma then follows from (3.1.2).  $\square$

For any  $1 \leq i \leq r+1$  and  $w \in \mathcal{D}_\delta \cap W_\mu$ , recalling (2.1.4) we introduce the following subset of  $W$ :

$$K_w^{(i)} = \left\{ \text{products of disjoint transpositions of the form } (j, k)_c \mid \begin{array}{l} j \in R_{3i-2}^\delta, k \in R_{3i-1}^\delta, (wg_2)^{-1}(k) < (wg_2)^{-1}(j) \end{array} \right\}. \quad (3.1.4)$$

We then define

$$K_w := \left\{ \prod_{i=1}^{r+1} \sigma^{(i)} \mid \sigma^{(i)} \in K_w^{(i)} \right\}. \quad (3.1.5)$$

Clearly we have  $\sigma^2 = 1$  for any  $\sigma \in K_w$ . For  $w \in \mathcal{D}_\delta \cap W_\mu$  and  $\sigma \in K_w$  we denote

$$n(\sigma) = \sum_{i=1}^{r+1} s_i, \quad \text{if } \sigma = \prod_{i=1}^{r+1} \prod_{l=1}^{s_i} (j_l^{(i)}, k_l^{(i)})_c. \quad (3.1.6)$$

We also set

$$h(w, \sigma) = |H(w, \sigma)|, \quad (3.1.7)$$

where

$$H(w, \sigma) = \bigcup_{i=1}^{r+1} \left\{ (j, k) \in R_{3i-2}^\delta \times R_{3i-1}^\delta \mid \begin{array}{l} (wg_2)^{-1}\sigma(j) > (wg_2)^{-1}(k), \\ (wg_2)^{-1}(j) > (wg_2)^{-1}\sigma(k) \end{array} \right\}. \quad (3.1.8)$$

**3.2. Multiplication formula for affine Hecke algebra.** The goal of this subsection is to establish the following formula for structure constants of the affine Hecke algebra  $\mathbf{H}$ . The formula will be applied in Section 4 to compute structure constants for affine Schur algebra  $\mathbf{S}_{n,d}^c$ . Recall  $\mathcal{D}_\lambda$  from (2.2.4),  $\mathcal{D}_{\lambda\mu}$  from (2.2.5),  $\kappa$  from (2.5.2),  $\delta(B)$  from (3.1.1),  $K_w$  from (3.1.5),  $n(\sigma)$  from (3.1.6), and  $h(w, \sigma)$  from (3.1.7).

**Theorem 3.3.** *Let  $B = \kappa(\lambda, g_1, \mu)$  be a tridiagonal matrix. Then, for any  $g_2 \in \mathcal{D}_{\mu\nu}$  and  $w \in \mathcal{D}_{\delta(B)} \cap W_\mu$ , we have*

$$T_{g_1} T_{wg_2} = \sum_{\sigma \in K_w} (v^2 - 1)^{n(\sigma)} v^{2h(w, \sigma)} T_{g_1 \sigma w g_2}. \quad (3.2.1)$$

*Proof.* Recall  $g_1 = \prod_{i=1}^{r+1} g_1^{(i)}$ ; see (3.1.2). It suffices to show that

$$T_{g_1^{(i)}} T_{wg_2} = \sum_{\sigma \in K_w^{(i)}} (v^2 - 1)^{n(\sigma)} v^{2h(w, \sigma)} T_{g_1^{(i)} \sigma wg_2} \quad (3.2.2)$$

for any  $1 \leq i \leq r+1$ . We shall continue to use the notations in Lemma 3.2 in this proof.

By Lemma 3.2, we have

$$T_{g_1^{(i)}} T_{wg_2} = (T_{s_{m+\beta}} \cdots T_{s_{m+2}} T_{s_{m+1}}) (T_{s_{m+\beta+1}} \cdots T_{s_{m+2}}) \cdots (T_{s_{m+\beta+\alpha-1}} \cdots T_{s_{m+\alpha}}) T_{wg_2}.$$

Let us denote

$$\{k \in R_{3i-1}^\delta | (wg_2)^{-1}(m+\alpha) > (wg_2)^{-1}(k)\} = \{m+\alpha+1, m+\alpha+2, \dots, m+\alpha+p\}.$$

We claim that

$$\begin{aligned} & (T_{s_{m+\beta+\alpha-1}} \cdots T_{s_{m+\alpha}}) T_{wg_2} \\ &= v^{2p} T_{s_{m+\beta+\alpha-1} \cdots s_{m+\alpha} wg_2} + \sum_{x=1}^p (v^2 - 1) v^{2(x-1)} T_{s_{m+\beta+\alpha-1} \cdots s_{m+\alpha} (m+\alpha, m+\alpha+x)_c wg_2}. \end{aligned} \quad (3.2.3)$$

We prove Equation (3.2.3) recursively. Firstly,

$$T_{s_{m+\alpha}} T_{wg_2} = \begin{cases} T_{s_{m+\alpha} wg_2}, & \text{if } (wg_2)^{-1}(m+\alpha) < (wg_2)^{-1}(m+\alpha+1); \\ v^2 T_{s_{m+\alpha} wg_2} + (v^2 - 1) T_{wg_2}, & \text{if } (wg_2)^{-1}(m+\alpha+1) < (wg_2)^{-1}(m+\alpha). \end{cases}$$

Suppose that we have shown, for  $c < p$ ,

$$\begin{aligned} & (T_{s_{m+\alpha+c-1}} \cdots T_{s_{m+\alpha}}) T_{wg_2} \\ &= v^{2c} T_{s_{m+\alpha+c-1} \cdots s_{m+\alpha} wg_2} + \sum_{x=1}^c (v^2 - 1) v^{2(x-1)} T_{s_{m+\alpha+c-1} \cdots s_{m+\alpha} (m+\alpha, m+\alpha+x)_c wg_2}. \end{aligned}$$

Note that

$$T_{s_{m+\alpha+c}} T_{s_{m+\alpha+c-1} \cdots s_{m+\alpha} wg_2} = v^2 T_{s_{m+\alpha+c} \cdots s_{m+\alpha} wg_2} + (v^2 - 1) T_{s_{m+\alpha+c} \cdots s_{m+\alpha} (m+\alpha, m+\alpha+c+1)_c wg_2}$$

thanks to

$$(s_{m+\alpha+c-1} \cdots s_{m+\alpha} wg_2)^{-1}(m+\alpha) > (s_{m+\alpha+c-1} \cdots s_{m+\alpha} wg_2)^{-1}(m+\alpha+c+1),$$

while

$$T_{s_{m+\alpha+c}} T_{s_{m+\alpha+c-1} \cdots s_{m+\alpha} (m+\alpha, m+\alpha+x)_c wg_2} = T_{s_{m+\alpha+c} \cdots s_{m+\alpha} (m+\alpha, m+\alpha+x)_c wg_2}$$

for any  $1 \leq x \leq c$ , thanks to

$$\begin{aligned} & (s_{m+\alpha+c-1} \cdots s_{m+\alpha} (m+\alpha, m+\alpha+x)_c wg_2)^{-1}(m+\alpha) \\ & > (s_{m+\alpha+c-1} \cdots s_{m+\alpha} (m+\alpha, m+\alpha+x)_c wg_2)^{-1}(m+\alpha+c+1). \end{aligned}$$

Therefore we obtain

$$\begin{aligned} & (T_{s_{m+\alpha+c}} \cdots T_{s_{m+\alpha}}) T_{wg_2} \\ &= v^{2(c+1)} T_{s_{m+\alpha+c} \cdots s_{m+\alpha} wg_2} + \sum_{x=1}^{c+1} (v^2 - 1) v^{2(x-1)} T_{s_{m+\alpha+c} \cdots s_{m+\alpha} (m+\alpha, m+\alpha+x)_c wg_2}. \end{aligned}$$

Suppose that we have obtained, for  $c \geq p$ ,

$$\begin{aligned} & (T_{s_{m+\alpha+c-1}} \cdots T_{s_{m+\alpha}}) T_{wg_2} \\ &= v^{2p} T_{s_{m+\alpha+c-1} \cdots s_{m+\alpha} wg_2} + \sum_{x=1}^p (v^2 - 1) v^{2(x-1)} T_{s_{m+\alpha+c-1} \cdots s_{m+\alpha} (m+\alpha, m+\alpha+x)_c wg_2}. \end{aligned}$$

Since

$$(s_{m+\alpha+c-1} \cdots s_{m+\alpha} wg_2)^{-1}(m+\alpha) < (s_{m+\alpha+c-1} \cdots s_{m+\alpha} wg_2)^{-1}(m+\alpha+c+1)$$

and

$$\begin{aligned} & (s_{m+\alpha+c-1} \cdots s_{m+\alpha} (m+\alpha, m+\alpha+x)_c)^{-1}(m+\alpha) \\ & < (s_{m+\alpha+c-1} \cdots s_{m+\alpha} (m+\alpha, m+\alpha+x)_c)^{-1}(m+\alpha+c+1), \end{aligned}$$

we have

$$(T_{s_{m+\alpha+c}} \cdots T_{s_{m+\alpha}}) T_{wg_2} = v^{2p} T_{s_{m+\alpha+c} \cdots s_{m+\alpha} wg_2} + \sum_{x=1}^p (v^2 - 1) v^{2(x-1)} T_{s_{m+\alpha+c} \cdots s_{m+\alpha} (m+\alpha, m+\alpha+x)_c wg_2}.$$

Thus (3.2.3) holds.

The power of  $v^2$  for each term on the right hand side of (3.2.3) is the number of elements  $k \in R_{3i-1}^\delta$  satisfying  $(wg_2)^{-1}\eta(m+\alpha) > (wg_2)^{-1}(k)$  where  $\eta = \mathbb{1}$  or  $(m+\alpha, m+\alpha+x)_c$ . The power of  $v^2 - 1$  for each term on the right hand side of (3.2.3) is the number of disjoint transpositions for  $\eta$ . In other words, (3.2.3) can be rewritten as follows

$$(T_{s_{m+\beta+\alpha-1}} \cdots T_{s_{m+\alpha}}) T_{wg_2} = \sum_{\substack{\eta=\mathbb{1}, (m+\alpha, m+\alpha+x)_c \\ x=1, 2, \dots, p}} (v^2 - 1)^{n(\eta)} v^{2h'(\eta)} T_{s_{m+\beta+\alpha-1} \cdots s_{m+\alpha} \eta wg_2} \quad (3.2.4)$$

where  $h'(\eta) := \left| \{(m+\alpha, k) \in R_{3i-2}^\delta \times R_{3i-1}^\delta \mid (wg_2)^{-1}\eta(m+\alpha) > (wg_2)^{-1}(k)\} \right|$ .

Replacing  $wg_2$  by  $\eta wg_2$ , we can obtain a similar formula for  $(T_{s_{m+\beta+\alpha-2}} \cdots T_{s_{m+\alpha-1}}) T_{\eta wg_2}$ . However, in case of  $\eta = (m+\alpha, m+\alpha+x)_c$ , we note that

$$\begin{aligned} & \{k \in R_{3i-1}^\delta \mid (\eta wg_2)^{-1}(k) < (\eta wg_2)^{-1}(m+\alpha-1)\} \\ &= \{k \in R_{3i-1}^\delta \mid (wg_2)^{-1}(k) < (wg_2)^{-1}(m+\alpha-1)\} \setminus \{m+\alpha+x\} \\ &= \{k \in R_{3i-1}^\delta \mid (wg_2)^{-1}(k) < (wg_2)^{-1}\eta(m+\alpha-1), (wg_2)^{-1}\eta(k) < (wg_2)^{-1}(m+\alpha-1)\}. \end{aligned}$$

Hence

$$\begin{aligned} & (T_{s_{m+\beta+\alpha-2}} \cdots T_{s_{m+\alpha-1}}) (T_{s_{m+\beta+\alpha-1}} \cdots T_{s_{m+\alpha}}) T_{wg_2} \\ &= \sum_{\zeta} (v^2 - 1)^{n(\zeta)} v^{2h'(\zeta)} T_{s_{m+\beta+\alpha-2} \cdots s_{m+\alpha-1} s_{m+\beta+\alpha-1} \cdots s_{m+\alpha} \zeta wg_2}, \end{aligned}$$

where  $\zeta$  runs over  $\mathbb{1}, (m+\alpha-1, k_1)_c, (m+\alpha, k_2)_c, (m+\alpha-1, k_1)_c(m+\alpha, k_2)_c, (k_1 \neq k_2)$  with  $wg_2^{-1}(m+\alpha-1) > wg_2^{-1}(k_1)$  and  $wg_2^{-1}(m+\alpha) > wg_2^{-1}(k_2)$ , and

$$h'(\zeta) = \left| \left\{ (j, k) \in R_{3i-2}^\delta \times R_{3i-1}^\delta \mid \begin{array}{l} j = m+\alpha \text{ or } m+\alpha-1, \\ (wg_2)^{-1}\zeta(j) > (wg_2)^{-1}(k), (wg_2)^{-1}(j) > (wg_2)^{-1}\zeta(k) \end{array} \right\} \right|.$$

Repeating this procedure, we have proved the formula (3.2.2). The theorem is proved.  $\square$

**Remark 3.4.** There is an intuitive explanation for  $h(w, \sigma)$  as follows. For each  $j \in R_{3i-2}^\delta$ , where  $1 \leq i \leq r+1$ , count the number of those elements  $k \in R_{3i-1}^\delta$  subject to Conditions (1)–(3) below: (1) there is no transposition  $(j', k)$ , ( $j < j' \in R_{3i-2}^\delta$ ), appearing in  $\sigma$ ; (2) the entry of  $A^P$  in which  $w^{-1}(k)$  lying is to the left of entry in which  $w^{-1}(j)$  lying (i.e.  $(wg_2)^{-1}(j) > (wg_2)^{-1}(k)$ ); (3)  $k < k'$  if  $(j, k')$  is a transposition appearing in  $\sigma$ . The sum of these numbers for all  $j \in R_{3i-2}^\delta$  and all  $1 \leq i \leq r+1$  is  $h(w, \sigma)$ .

**Corollary 3.5.** For any  $g_1 \in \mathcal{D}_{\lambda\mu}$ ,  $g_2 \in \mathcal{D}_{\mu\nu}$ ,  $w \in \mathcal{D}_\delta \cap W_\mu$  and  $\sigma \in K_w$ , we have

$$\ell(g_1) + \ell(w) + \ell(g_2) = \ell(g_1 \sigma w g_2) + n(\sigma) + 2h(w, \sigma). \quad (3.2.5)$$

*Proof.* Follows from (3.2.1).  $\square$

### 3.3. An example.

**Example 3.6.** Let  $r = 2, n = 6, d = 8$  and  $D = 18$ . Let  $B = E^{00} + 2 \sum_{1 \leq i, j \leq 2} E_\theta^{ij} + E^{33}$  and

$A = E^{00} + \sum_{i=1}^2 \sum_{j=1}^4 E_\theta^{ij} + E^{33}$ . Namely,

$$B = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & 2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2 & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{bmatrix},$$

where dashed stripes denote the 0th column/row. Here we use a two-by-four submatrix for short when there is no ambiguity. That is,

$$B = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Thus,  $g_1 = g^{\text{std}} \left( \begin{bmatrix} 1234 & & \\ 5678 & & \end{bmatrix} \right) = [1, 2, 5, 6, 3, 4, 7, 8]_c$ ; see (2.1.3). On the other hand, we have

$$\begin{array}{c|cccccccccc} i & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline R_i^\delta & \{0\} & & [1..2] & [3..4] & [5..6] & [7..8] & & & & \{9\} \end{array}$$

That is,  $g_1 = g_1^{(2)} = (s_4 s_3)(s_5 s_4)$  with  $m = \alpha = \beta = 2$ .

Also,  $g_2 = g^{\text{std}} \left( \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \right) = [1, 5, 2, 6, 3, 7, 4, 8]_c$ .

Now write  $T \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix} = T_x$  where  $x = g^{\text{std}} \left( \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix} \right)$  for short. We have

$$\begin{aligned} T_4 T \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} &= v^2 T \begin{bmatrix} 1 & 2 & 3 & 5 \\ 4 & 6 & 7 & 8 \end{bmatrix} + (v^2 - 1) T \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}, \\ T_5 T_4 T \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} &= v^2 T \begin{bmatrix} 1 & 2 & 3 & 6 \\ 4 & 5 & 7 & 8 \end{bmatrix} + (v^2 - 1) \left( v^2 T \begin{bmatrix} 1 & 2 & 3 & 5 \\ 4 & 6 & 7 & 8 \end{bmatrix} + T \begin{bmatrix} 1 & 2 & 3 & 4 \\ 6 & 5 & 7 & 8 \end{bmatrix} \right). \end{aligned}$$

If  $w = \mathbb{1}$ , we have

$$K_w^{(1)} = K_w^{(3)} = \{\mathbb{1}\}, \quad K_w^{(2)} = K_w = \{\mathbb{1}, (3, 5)_c, (3, 6)_c, (4, 5)_c, (4, 6)_c, (3, 5)_c(4, 6)_c, (3, 6)_c(4, 5)_c\},$$

and  $T_{g_1}T_{wg_2} = \sum (v^2 - 1)^{n(\sigma)} v^{2h(\sigma)} T_{g_1\sigma wg_2}$  with

$T_{g_1\sigma wg_2}$	$\sigma$	$n(\sigma)$	$h(\sigma)$								
$T$ <table border="1"><tr><td></td><td></td><td>5</td><td>6</td></tr><tr><td>3</td><td>4</td><td></td><td></td></tr></table>			5	6	3	4			$\mathbb{1}$	0	4
		5	6								
3	4										
$T$ <table border="1"><tr><td></td><td></td><td>4</td><td>6</td></tr><tr><td>3</td><td>5</td><td></td><td></td></tr></table>			4	6	3	5			$(3, 6)_c$	1	3
		4	6								
3	5										
$T$ <table border="1"><tr><td></td><td></td><td>3</td><td>6</td></tr><tr><td>5</td><td>4</td><td></td><td></td></tr></table>			3	6	5	4			$(3, 5)_c$	1	2
		3	6								
5	4										
$T$ <table border="1"><tr><td></td><td></td><td>5</td><td>4</td></tr><tr><td>3</td><td>6</td><td></td><td></td></tr></table>			5	4	3	6			$(4, 6)_c$	1	2
		5	4								
3	6										
$T$ <table border="1"><tr><td></td><td></td><td>3</td><td>4</td></tr><tr><td>5</td><td>6</td><td></td><td></td></tr></table>			3	4	5	6			$(3, 5)_c(4, 6)_c$	2	1
		3	4								
5	6										
$T$ <table border="1"><tr><td></td><td></td><td>5</td><td>3</td></tr><tr><td>6</td><td>4</td><td></td><td></td></tr></table>			5	3	6	4			$(4, 5)_c$	1	1
		5	3								
6	4										
$T$ <table border="1"><tr><td></td><td></td><td>4</td><td>3</td></tr><tr><td>6</td><td>5</td><td></td><td></td></tr></table>			4	3	6	5			$(3, 6)_c(4, 5)_c$	2	0
		4	3								
6	5										

#### 4. MULTIPLICATION FORMULA FOR AFFINE SCHUR ALGEBRA

This section is devoted to the multiplication formula with tridiagonal generators. An essential idea in the proof is to identify the basis element  $e_A$  with its corresponding “higher-level” matrix with entries being subsets of  $\mathbb{Z}$  by (2.4.2). We also provide two special cases of the multiplication formula that are analogous to the multiplication formulas with semisimple generators in affine type A, and with Chevalley generators as in finite type B/C.

**4.1. A map  $\varphi$ .** Recall  $T_\theta$  from (2.3.7). We introduce an *entry-wise partial order*, denoted by  $\leq_e$ , on  $\Theta_n$  by a matrix-entry-wise comparison:

$$(a_{ij}) \leq_e (b_{ij}) \Leftrightarrow a_{ij} \leq b_{ij} (\forall i, j). \quad (4.1.1)$$

For  $A, B \in \Xi_n$ , we set

$$\Theta_{B,A} = \{T \in \Theta_n \mid T_\theta \leq_e A, \text{ row}_a(T)_i = b_{i-1,i} \text{ for all } i\}. \quad (4.1.2)$$

Fix  $\lambda, \mu, \nu \in \Lambda$ ,  $g_1 \in \mathcal{D}_{\lambda\mu}$ , and  $g_2 \in \mathcal{D}_{\mu\nu}$ . Let  $B = \kappa(\lambda, g_1, \mu)$  and  $A = \kappa(\mu, g_2, \nu)$ . Recall  $\delta = \delta(B)$  from (3.1.1). For  $w \in W$ , we associate a matrix  $\varphi(w)$  whose  $(i, j)$ th entry of the matrix  $\varphi(w)$  is given by

$$\varphi(w)_{ij} = |R_{3i-1}^\delta \cap wg_2 R_j^\nu|. \quad (4.1.3)$$

**Lemma 4.1.** *We have a surjective map  $\varphi : \mathcal{D}_\delta \cap W_\mu \longrightarrow \Theta_{B,A}$ ,  $w \mapsto \varphi(w)$ .*

*Proof.* We first note that  $\varphi(w) \in \Theta_n$ . For any  $w \in \mathcal{D}_\delta \cap W_\mu$  and any  $i$ , we have

$$\text{row}_a(\varphi(w))_i = |R_{3i-1}^\delta| = b_{i-1,i}.$$

Next we check that  $\varphi(w) \leq_e A$ . Indeed, by Lemma 3.1, we have

$$\begin{aligned} \varphi(w)_{\theta,ij} &= |R_{3i-1}^\delta \cap wg_2R_j^\nu| + |R_{3(-i)-1}^\delta \cap wg_2R_{-j}^\nu| \\ &= |(R_{3i-1}^\delta \cup R_{3i+1}^\delta) \cap wg_2R_j^\nu| \\ &\leq |R_i^\mu \cap wg_2R_j^\nu| \\ &= |w^{-1}R_i^\mu \cap g_2R_j^\nu| = |R_i^\mu \cap g_2R_j^\nu| = a_{ij}. \end{aligned}$$

Hence we have proved  $\varphi(w) \in \Theta_{B,A}$ .

To show that  $\varphi$  is surjective, for any  $T = (t_{ij}) \in \Theta_{B,A}$  we shall construct an element  $w_{A,T} \in \varphi^{-1}(T)$  as follows. Recall  $A^P$  from (2.4.2). For all  $i, j$ , we set

$$\begin{aligned} \mathcal{T}_{ij}^- &= \text{subset of } (A^P)_{ij} \text{ consisting of the smallest } t_{ij} \text{ elements,} \\ \mathcal{T}_{ij}^+ &= \text{subset of } (A^P)_{ij} \text{ consisting of the largest } t_{-i,-j} \text{ elements,} \\ \mathcal{T}_{ij}^0 &= (A^P)_{ij} - \mathcal{T}_{ij}^+ - \mathcal{T}_{ij}^-. \end{aligned}$$

Note that  $\sum_{j \in \mathbb{Z}} |\mathcal{T}_{ij}^-| = \text{row}_a(T)_i = |R_{3i-1}^\delta|$ ,  $\sum_{j \in \mathbb{Z}} |\mathcal{T}_{ij}^+| = \text{row}_a(T)_{-i} = |R_{3i+1}^\delta|$ . There is a unique element  $w_{A,T} = \prod_{i=0}^{r+1} w_{A,T}^{(i)} \in \mathcal{D}_\delta \cap W_\mu$  where  $w_{A,T}^{(i)} \in \text{Perm}(R_i^\mu)$  is determined by

$$w_{A,T}^{(i)}(x) \in \begin{cases} R_{3i-1}^\delta, & \text{if } x \in \bigcup_j \mathcal{T}_{ij}^- \\ R_{3i+1}^\delta, & \text{if } x \in \bigcup_j \mathcal{T}_{ij}^+ \\ R_{3i}^\delta, & \text{otherwise.} \end{cases} \quad (4.1.4)$$

The uniqueness follows from that  $w_{A,T}^{-1}$  is order-preserving on each  $R_j^\delta$  (cf. Lemma 2.3).

One verifies by construction that  $\varphi(w_{A,T}) = T$ . The lemma is proved.  $\square$

**Lemma 4.2.** *For  $T \in \Theta_{B,A}$ , the element  $w_{A,T}$  determined by (4.1.4) is the minimal length element in  $\varphi^{-1}(T)$ . Moreover, its length is*

$$\begin{aligned} \ell(w_{A,T}) &= \sum_{\substack{1 \leq i \leq r \\ j \in \mathbb{Z}}} \left( t_{ij} \sum_{k < j} (A - T)_{ik} + t_{-i,-j} \sum_{k > j} (A - T)_{ik} \right) \\ &+ \sum_{\substack{k < j \leq 0 \\ \text{or} \\ -k \geq j > 0}} t_{0j} (A - T)_{0k} + \sum_{|k| < j} t_{0j} (A - T)_{0k} - \sum_{j > 0} \binom{t_{0j}}{2} \\ &+ \sum_{\substack{k < j \leq r+1 \\ \text{or} \\ n-k \geq j > r+1}} t_{r+1,j} (A - T)_{r+1,k} + \sum_{\substack{j > r+1, \\ |k-r-1| < j}} t_{r+1,j} (A - T)_{r+1,k} - \sum_{j > r+1} \binom{t_{r+1,j}}{2}. \end{aligned} \quad (4.1.5)$$

*Proof.* It follows by construction that  $w_{A,T}$  is the minimal length element in  $\varphi^{-1}(T)$ .

The permutation  $w_{A,T}$  shifts elements in  $\bigcup_{j \in \mathbb{Z}} \mathcal{T}_{ij}^-$  to the front of elements in the union  $\bigcup_{j \in \mathbb{Z}} (A^P)_{ij} \setminus \mathcal{T}_{ij}^-$ , and shifts elements in  $\bigcup_{j \in \mathbb{Z}} \mathcal{T}_{ij}^+$  to the back of  $\bigcup_{j \in \mathbb{Z}} (A^P)_{ij} \setminus \mathcal{T}_{ij}^+$ . Note that  $|\mathcal{T}_{ij}^-| = t_{ij}$  and  $|\mathcal{T}_{ij}^+| = t_{-i,-j}$ .

For  $1 \leq i \leq r$ , we first count that there are  $\sum_{j \in \mathbb{Z}} (t_{ij} \sum_{k < j} (a_{ik} - t_{ik}))$  elements in total crossed by  $\bigcup_{j \in \mathbb{Z}} \mathcal{T}_{ij}^-$ , and then there are  $\sum_{j \in \mathbb{Z}} (t_{-i, -j} \sum_{k > j} (a_{ik} - t_{ik} - t_{-i, -k}))$  elements in total crossed by  $\bigcup_{j \in \mathbb{Z}} \mathcal{T}_{ij}^+$ . This accounts for the sum on the first line of the RHS of (4.1.5).

When we move elements in  $\mathcal{T}_{0j}^-$ , their opposite elements in  $\mathcal{T}_{0, -j}^+$  are moved automatically in a symmetric way. Thus there are  $\sum_{k < j \leq 0} t_{0j} (a_{0k} - t_{0k})$  elements in total crossed by  $\bigcup_{j \leq 0} \mathcal{T}_{0j}^-$ . And for each  $j > 0$ , there are  $t_{0j} \sum_{k < -j} (a_{0k} - t_{0k}) + (t_{0j} (a_{0, -j} - t_{0, -j}) - \sum_{s=1}^{t_{0j}-1} s) + t_{0j} \sum_{|k| < j} (a_{0k} - t_{0k} - t_{0, -k})$  elements in total crossed by  $\mathcal{T}_{0j}^-$ . Adding these contributions gives us the second line on the RHS of (4.1.5). Similarly, we obtain the third line of the RHS of (4.1.5) by considering  $\mathcal{T}_{r+1, j}^\pm$ .  $\square$

**Lemma 4.3.** *Let  $T \in \Theta_{B, A}$ , we have*

$$\sum_{w \in \varphi^{-1}(T)} v^{2\ell(w)} = v^{2\ell(w_{A, T})} \frac{[A]_{\mathfrak{c}}!}{[A - T\theta]_{\mathfrak{c}}! [T]!}. \quad (4.1.6)$$

*Proof.* Any permutation  $w \in \varphi^{-1}(T)$  is determined by which  $t_{ij}$  and  $t_{-i, -j}$  elements in  $(A^P)_{i, j}$  is moved to left and right for all  $(i, j) \in I^+$  (see (2.3.12)), respectively. Hence  $\sum_{w \in \varphi^{-1}(T)} v^{2\ell(w)}$ , which is the generating function in  $v$  of counting inversions for  $\varphi^{-1}(T)$ , is a product of the generating functions of counting inversions locally for each  $(i, j) \in I^+$ .

Recall  $I^+ = I_{\mathfrak{a}}^+ \cup \{(0, 0), (r+1, r+1)\}$  from (2.4.1). Let us determine these generating functions of local inversion countings one by one. For each entry  $(i, j) \in I_{\mathfrak{a}}^+$ , it is given by a standard recipe  $\frac{[a_{ij}]!}{[t_{ij}]! [t_{-i, -j}]! [a_{ij} - (t_{\theta})_{ij}]!}$ .

Let  $k = 0$  or  $r+1$ . Recall  $a'_{kk} = \frac{a_{kk}-1}{2}$  from (2.3.10). The generating function of counting inversions locally at the entry  $(k, k) \in I^+$  is given by

$$\begin{aligned} \sum_{x+y=t_{kk}} \begin{bmatrix} a'_{kk} \\ x \end{bmatrix} \begin{bmatrix} a'_{kk} - x \\ y \end{bmatrix} (v^2)^{\frac{x(x+1)}{2} + x(a'_{kk} - t_{kk})} \\ = \begin{bmatrix} a'_{kk} \\ t_{kk} \end{bmatrix} \sum_{x=0}^{t_{kk}} \begin{bmatrix} t_{kk} \\ x \end{bmatrix} v^{x(x-1)} (v^{a'_{kk}+1-t_{kk}})^{2x} \\ \stackrel{(\diamond)}{=} \begin{bmatrix} a'_{kk} \\ t_{kk} \end{bmatrix} \prod_{i=1}^{t_{kk}} (1 + v^{2(i-1)} v^{2(a'_{kk}-t_{kk}+1)}) = \frac{[a_{kk}]_{\mathfrak{c}}!}{[a_{kk} - 2t_{kk}]_{\mathfrak{c}}! [t_{kk}]!}, \end{aligned}$$

where  $(\diamond)$  uses the quantum binomial theorem  $\sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} v^{r(r-1)} x^r = \prod_{k=0}^{n-1} (1 + v^{2k} x)$ .

Summarizing, we have obtained

$$\begin{aligned} \sum_{w \in \varphi^{-1}(T)} v^{2\ell(w)} &= v^{2\ell(w_{A, T})} \prod_{(i, j) \in I_{\mathfrak{a}}^+} \frac{[a_{ij}]!}{[t_{ij}]! [t_{-i, -j}]! [a_{ij} - (t_{\theta})_{ij}]!} \prod_{k=0, r+1} \frac{[a_{kk}]_{\mathfrak{c}}!}{[a_{kk} - 2t_{kk}]_{\mathfrak{c}}! [t_{kk}]!} \\ &= v^{2\ell(w_{A, T})} \frac{[A]_{\mathfrak{c}}!}{[A - T\theta]_{\mathfrak{c}}! [T]!}. \end{aligned}$$

The lemma follows.  $\square$



Recall the subset  $K_w \subset W$  from (3.1.4)–(3.1.5) for  $w \in \mathcal{D}_\delta \cap W_\mu$ . Also recall  $h(w, \sigma)$  from (3.1.7) and  $H(w, \sigma)$  from (3.1.8), for  $w \in \mathcal{D}_\delta \cap W_\mu$  and  $\sigma \in K_w$ . Recall further from (4.1) the map  $\varphi : \mathcal{D}_\delta \cap W_\mu \rightarrow \Theta_{B,A}$ .

**Lemma 4.4.** *Let  $w_1, w_2 \in \mathcal{D}_\delta \cap W_\mu$ . If  $\varphi(w_1) = \varphi(w_2)$ , then  $K_{w_1} = K_{w_2}$  and  $H(w_1, \sigma) = H(w_2, \sigma)$  for any  $\sigma \in K_{w_1} (= K_{w_2})$ ; in particular, we have  $h(w_1, \sigma) = h(w_2, \sigma)$ .*

*Proof.* It follows from  $\varphi(w_1) = \varphi(w_2)$  that  $w_1^{-1}(x)$  and  $w_2^{-1}(x)$  lie in the same entry of  $A^P$ , for any  $x \in R_{3i-2}^\delta \cup R_{3i-1}^\delta$ . Thus for any  $j \in R_{3i-2}^\delta$  and  $k \in R_{3i-1}^\delta$ , we have  $g_2^{-1}w_1^{-1}(k) < g_2^{-1}w_1^{-1}(j)$  if and only if  $g_2^{-1}w_2^{-1}(k) < g_2^{-1}w_2^{-1}(j)$ . So  $K_{w_1} = K_{w_2}$  and  $H(w_1, \sigma) = H(w_2, \sigma)$  by the definition (3.1.5) and (3.1.8).  $\square$

Thanks to Lemma 4.4, for  $T \in \Theta_{B,A}$ , we can define

$$K(T) = K_w \quad \text{for some (or for all) } w \in \varphi^{-1}(T), \quad (4.1.7)$$

and further define, for  $\sigma \in K(T)$ ,

$$h(T, \sigma) = h(w, \sigma) \quad \text{for some (or for all) } w \in \varphi^{-1}(T). \quad (4.1.8)$$

**4.2. Algebraic combinatorics for  $\mathbf{S}_{n,d}^c$ .** For any  $w \in \mathcal{D}_\delta \cap W_\mu$  and  $\sigma \in K_w$ , we recall from (2.6.7) that  $g_1 \sigma w g_2 = w_\lambda^{(\sigma)} y^{(w, \sigma)} w_\nu^{(\sigma)}$  and from (2.6.8) that  $A^{(w, \sigma)} = (a_{ij}^{(w, \sigma)}) = \kappa(\lambda, y^{(w, \sigma)}, \nu)$ .

Recall  $\sigma = \prod_{i=1}^{r+1} \sigma^{(i)} \in K_w$  with  $\sigma^{(i)} \in K_w^{(i)}$ . Let us fix the product  $\sigma^{(i)} = \prod_{l=1}^{s_i} (j_l^{(i)}, k_l^{(i)})_c$  by requiring

$$j_1^{(i)} < j_2^{(i)} < \cdots < j_{s_i}^{(i)}. \quad (4.2.1)$$

We further define  $s_{-i} = s_{i+1}$  for  $0 \leq i \leq r$  and

$$j_l^{(-i)} = k_{s_{i+1}-l+1}^{(i+1)}, \quad k_l^{(-i)} = j_{s_{i+1}-l+1}^{(i+1)} \quad \text{for } 0 \leq i \leq r, 1 \leq l \leq s_i. \quad (4.2.2)$$

Hence the permutations  $\sigma^{(-i)} = \prod_{l=1}^{s_{-i}} (j_l^{(-i)}, k_l^{(-i)})_c$  for  $0 \leq i \leq r$  satisfy (4.2.1) as well. For  $w \in \mathcal{D}_\delta \cap W_\mu$ , we define a map

$$\psi_w : K_w \longrightarrow \Theta_n, \quad (4.2.3)$$

$$\psi_w(\sigma)_{ij} = |R_{3i-1}^\delta \cap \sigma(R_{3i-2}^\delta) \cap w g_2 R_j^\nu| = |\{k_l^{(i)}\}_{l=1}^{s_i} \cap w g_2 R_j^\nu|.$$

For any  $\mathbb{Z} \times \mathbb{Z}$  matrix  $T = (t_{ij})$ , we set

$$\widehat{T} = (\widehat{t}_{ij}), \quad \text{where } \widehat{t}_{ij} = t_{i+1, j}. \quad (4.2.4)$$

Assume that  $S = \psi_w(\sigma)$  for some  $w \in \mathcal{D}_\delta \cap W_\mu$  and  $\sigma \in K_w$ . By (4.2.2) we have

$$\begin{aligned} \widehat{s}_{ij} &= |\{k_l^{(i+1)}\}_l \cap w g_2 R_j^\nu|, \\ s_{-i, -j} &= |\{j_l^{(i+1)}\}_l \cap w g_2 R_j^\nu|, \\ (\widehat{s})_{-i, -j} &= |\{j_l^{(i)}\}_l \cap w g_2 R_j^\nu|. \end{aligned} \quad (4.2.5)$$

For any matrix  $S = (s_{ij})$ , denote by  $S^\dagger = (s_{ij}^\dagger)$  the matrix obtained by rotating the matrix  $\widehat{S}$  by 180 degrees, namely,

$$s_{ij}^\dagger = s_{1-i, -j} = \widehat{s}_{-i, -j}. \quad (4.2.6)$$

For  $T \in \Theta_{B,A}$  (cf. (4.1.2)), we set

$$\Gamma_T = \{S \in \Theta_n \mid S \leq_e T, \text{row}_a(S) = \text{row}_a(S^\dagger)\}. \quad (4.2.7)$$

**Lemma 4.5.** *For  $w \in \mathcal{D}_\delta \cap W_\mu$ , we have  $\psi_w(K_w) \subset \Gamma_{\varphi(w)}$ .*

*Proof.* For each  $\sigma \in K_w$ , it follows from (4.2.2) that  $\text{row}_a(\psi_w(\sigma)) = \text{row}_a(\psi_w(\sigma)^\dagger)$ . Also, by Lemma 4.1 we have

$$|\{k_1^{(i)}, k_2^{(i)}, \dots, k_{s_i}^{(i)}\} \cap wg_2 R_j^\nu| \leq |R_{3i-1}^\delta \cap wg_2 R_j^\nu| = \varphi(w)_{ij},$$

and hence  $\psi_w(\sigma) \leq T$ . □

For  $T \in \Theta_{B,A}$  (4.1.2) and  $S \in \Gamma_T$  (4.2.7), we set

$$A^{(T-S)} = A - (T - S)_\theta + (\widehat{T - S})_\theta, \quad \text{and } A^{(T)} = A^{(T-0)}. \quad (4.2.8)$$

Recall  $A^{(w,\sigma)}$  from (2.6.8).

**Lemma 4.6.** *For  $w \in \mathcal{D}_\delta \cap W_\mu$  and  $\sigma \in K_w$ , we have*

$$A^{(w,\sigma)} = A^{(T-S)}, \quad (4.2.9)$$

where  $T = \varphi(w)$  and  $S = \psi_w(\sigma)$ ; cf. (4.1.3) and (4.2.3).

*Proof.* By the definitions (4.1.3) and (4.2.3), we have

$$\begin{aligned} (T - S)_{ij} &= |(R_{3i-1}^\delta - \sigma(R_{3i-2}^\delta)) \cap wg_2 R_j^\nu|, \\ (T - S)_{-i,-j} &= |(R_{3i+1}^\delta - \sigma(R_{3i+2}^\delta)) \cap wg_2 R_j^\nu|, \\ (\widehat{T - S})_{ij} &= |(R_{3i+2}^\delta - \sigma(R_{3i+1}^\delta)) \cap wg_2 R_j^\nu|, \\ (\widehat{T - S})_{-i,-j} &= |(R_{3i-2}^\delta - \sigma(R_{3i-1}^\delta)) \cap wg_2 R_j^\nu|. \end{aligned} \quad (4.2.10)$$

Recall from Lemma 3.1 that  $R_i^\mu = R_{3i-1}^\delta \cup R_{3i}^\delta \cup R_{3i+1}^\delta$ , and hence

$$\begin{aligned} a_{ij} &= |R_i^\mu \cap wg_2 R_j^\nu| \\ &= |R_{3i-1}^\delta \cap wg_2 R_j^\nu| + |R_{3i}^\delta \cap wg_2 R_j^\nu| + |R_{3i+1}^\delta \cap wg_2 R_j^\nu|. \end{aligned} \quad (4.2.11)$$

Again by Lemma 3.1, we have  $g_1^{-1} R_i^\lambda = R_{3i-2}^\delta \cup R_{3i}^\delta \cup R_{3i+2}^\delta$ . Therefore, by using (4.2.10)–(4.2.11), we obtain

$$\begin{aligned} a_{ij}^{(w,\sigma)} &= |R_i^\lambda \cap g_1 \sigma wg_2 R_j^\nu| = |\sigma g_1^{-1} R_i^\lambda \cap wg_2 R_j^\nu| \\ &= |\sigma(R_{3i-2}^\delta) \cap wg_2 R_j^\nu| + |R_{3i}^\delta \cap wg_2 R_j^\nu| + |\sigma(R_{3i+2}^\delta) \cap wg_2 R_j^\nu| \\ &= a_{ij} - (T - S)_{\theta,ij} + (\widehat{T - S})_{\theta,ij}. \end{aligned}$$

The lemma is proved. □

**Example 4.7.** Retain the notation as in Example 3.6. The matrices in  $\Theta_{B,A}$  have zero rows except for the -1st and 2nd (mod  $n$ ) rows, and hence  $\Theta_{B,A} = \{T_x^{(-1)} + T_y^{(2)} \mid 1 \leq x, y \leq 6\}$ , where

$$\begin{aligned} \{T_x^{(-1)} \mid 1 \leq x \leq 6\} &= \{E^{-1,j} + E^{-1,k} \mid -4 \leq j < k \leq -1\}, \\ \{T_y^{(2)} \mid 1 \leq y \leq 6\} &= \{E^{2,j} + E^{2,k} \mid 1 \leq j < k \leq 4\}. \end{aligned}$$

For any  $\mathbb{Z} \times \mathbb{Z}$  matrix  $M = (m_{ij})$ , recalling (2.3.6)–(2.3.7), we introduce another short-hand notation

$$M_a = \sum_{(i,j) \in I^+} m_{\theta,ij} E^{ij}.$$

In particular, for  $T = E^{-1,-4} + E^{-1,-3} + E^{21} + E^{22} \in \Theta_{B,A}$ , we have

$$T_a = \begin{array}{|c|c|c|c|} \hline 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 \\ \hline \end{array}.$$

In this case we have  $\varphi^{-1}(T) = \{\mathbb{1}\}$ . Moreover, by Example 3.6, we have

$$K(T) = \{\mathbb{1}, (3, 5)_c, (3, 6)_c, (4, 5)_c, (4, 6)_c, (3, 5)_c(4, 6)_c, (3, 6)_c(4, 5)_c\}.$$

The complete list of  $S \in \Gamma_T$  is given by

$S$	$S_a$	$\psi_w^{-1}(S)$
0	$\begin{array}{ c c c c } \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array}$	$\{\mathbb{1}\}$
$E^{-1,-3} + E^{21}$	$\begin{array}{ c c c c } \hline 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ \hline \end{array}$	$\{(3, 5)_c\}$
$E^{-1,-3} + E^{22}$	$\begin{array}{ c c c c } \hline 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline \end{array}$	$\{(3, 6)_c\}$
$E^{-1,-4} + E^{21}$	$\begin{array}{ c c c c } \hline 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ \hline \end{array}$	$\{(4, 5)_c\}$
$E^{-1,-4} + E^{22}$	$\begin{array}{ c c c c } \hline 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \\ \hline \end{array}$	$\{(4, 6)_c\}$
$T$	$\begin{array}{ c c c c } \hline 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 \\ \hline \end{array}$	$\{(3, 5)_c(4, 6)_c, (3, 6)_c(4, 5)_c\}$

We define an element

$$\sigma_{w,S} = \prod_{i=1}^{r+1} \prod_{l=1}^{s_i} (j_l^{(i)}, k_l^{(i)})_c \in \psi_w^{-1}(S) \quad (4.2.12)$$

satisfying Conditions (S1)–(S2) below:

(S1)  $k_1^{(i)} < k_2^{(i)} < \dots < k_{s_i}^{(i)}, \forall i$ ;

(S2)  $w^{-1}(\{k_l^{(i)}\}_l) \cap g_2 R_j^\nu$  consists of the largest  $s_{ij}$  elements in  $w^{-1} R_{3i-1}^\delta \cap g_2 R_j^\nu, \forall i$ .

It follows from (4.2.1) that Conditions (S1)–(S2) together imply Condition (S3) below:

(S3)  $w^{-1}(\{j_l^{(i)}\}_l) \cap g_2 R_j^\nu$  consists of the smallest  $s_{ij}$  elements in  $w^{-1} R_{3i-2}^\delta \cap g_2 R_j^\nu, \forall i$ .

For  $S \in \Gamma_T$ , recall  $S^\dagger$  from (4.2.6) and set

$$\llbracket S \rrbracket = \prod_{i=1}^{r+1} \llbracket S \rrbracket_i, \quad (4.2.13)$$

where

$$\llbracket S \rrbracket_i = \prod_{j \in \mathbb{Z}} \left[ \begin{array}{c} \sum_{k \leq j} (S - S^\dagger)_{ik} \\ s_{i,j+1}^\dagger \end{array} \right] [s_{i,j+1}^\dagger]!. \quad (4.2.14)$$

Each  $\llbracket S \rrbracket_i$  counts the “quantum number” of pairs  $(x, y)$  in the following sense:

- (1) The element  $x$  contributes to the  $i$ th row of  $S$ . That is,  $x \in \{k_l^{(i)}\}_{l=1}^{s_i}$ ;
- (2) The element  $y$  contributes to the  $i$ th row of  $S^\dagger$ . That is,  $y \in \{j_l^{(i)}\}_{l=1}^{s_i}$ ;
- (3) The element  $x$  is “to the left” of  $y$  as elements in  $A^\mathcal{P}$ .

Recall  $\psi_w : K_w \rightarrow \Theta_n$  from (4.2.3) and  $\sigma_{w,S}$  from (4.2.12).

**Lemma 4.8.** *Let  $T \in \Theta_{B,A}$  and  $S \in \Gamma_T$ . For any  $w \in \varphi^{-1}(T)$ , we have*

$$\sum_{\sigma \in \psi_w^{-1}(S)} v^{-2h(w,\sigma)} = v^{-2h(w,\sigma_{w,S})} \left[ \begin{matrix} T \\ S \end{matrix} \right] \llbracket S \rrbracket. \quad (4.2.15)$$

*Proof.* Let  $s_i = \text{row}_a(S)_i = \text{row}_a(S^\dagger)_i$  for all  $i$ . By definition, each  $\sigma \in \psi_w^{-1}(S)$  can be reconstructed by the following steps:

- (1) For  $1 \leq i \leq r+1, j \in \mathbb{Z}$ , choose  $s_{i,j}^\dagger$  elements from the set  $R_{3i-2}^\delta \cap wg_2 R_j^\nu$ .
- (2) Let  $j_\sigma = \{j_1^{(i)}, j_2^{(i)}, \dots, j_{s_i}^{(i)}\}$  be the set of elements chosen from  $\bigcup_{j \in \mathbb{Z}} R_{3i-2}^\delta \cap wg_2 R_j^\nu$  such that  $j_1^{(i)} < j_2^{(i)} < \dots < j_{s_i}^{(i)}$ .
- (3) For  $1 \leq i \leq r+1, j \in \mathbb{Z}$ , choose  $s_{i,j}$  elements from the set  $R_{3i-1}^\delta \cap wg_2 R_j^\nu$ .
- (4) Let  $k_\sigma = \{k_1^{(i)}, k_2^{(i)}, \dots, k_{s_i}^{(i)}\}$  be the set of elements chosen from  $\bigcup_{j \in \mathbb{Z}} R_{3i-1}^\delta \cap wg_2 R_j^\nu$  such that

$$(wg_2)^{-1}(j_s^{(i)}) > (wg_2)^{-1}(k_s^{(i)}), \quad \text{for } s = 1, 2, \dots, s_i.$$

(Note that we do not impose that  $k_1^{(i)} < k_2^{(i)} < \dots < k_{s_i}^{(i)}$ .)

- (5) Set  $\sigma = \prod_{i=1}^{r+1} (j_1^{(i)}, k_1^{(i)})_c \cdots (j_{s_i}^{(i)}, k_{s_i}^{(i)})_c$ .

For those  $\sigma \in \psi_w^{-1}(S)$  having the same  $k_\sigma$  (say  $k_\sigma = K$ ), we pick a representative

$$\sigma^{\triangleleft K} = \prod_{i=1}^{r+1} (j_1^{(i)}, k_1^{(i)})_c \cdots (j_{s_i}^{(i)}, k_{s_i}^{(i)})_c$$

such that  $k_1^{(i)} < k_2^{(i)} < \dots < k_{s_i}^{(i)}$ . We claim that

$$\sum_{\sigma \in K_w, k_\sigma = K} v^{-2h(w,\sigma)} = \llbracket S \rrbracket v^{-2h(w,\sigma^{\triangleleft K})}. \quad (4.2.16)$$

We prove (4.2.16). Indeed, any  $\sigma \in K_w$  with  $k_\sigma = K$  must be of the form

$$\sigma = \prod_{i=1}^{r+1} (j_1^{(i)}, k_{\tau_i(1)}^{(i)})_c \cdots (j_{s_i}^{(i)}, k_{\tau_i(s_i)}^{(i)})_c$$

for some  $\tau_i \in K'_i$  ( $1 \leq i \leq r+1$ ), where

$$K'_i = \left\{ \tau \in \text{Perm}([1..s_i]) \mid (wg_2)^{-1}(j_s^{(i)}) > (wg_2)^{-1}(k_{\tau(s)}^{(i)}) \text{ for } 1 \leq s \leq s_i \right\}.$$

By a detailed calculation, for each  $i$  we have

$$\sum_{\tau \in K'_i} v^{2\ell(\tau)} = \prod_{j \in \mathbb{Z}} \left[ \begin{matrix} \sum_{l \leq j} (S - S^\dagger)_{il} \\ s_{i,j+1}^\dagger \end{matrix} \right] [s_{i,j+1}^\dagger]! = \llbracket S \rrbracket_i.$$

Now (4.2.16) follows by computing

$$\sum_{\sigma \in K_w, k_\sigma = K} v^{-2h(w,\sigma)} = \prod_{i=1}^{r+1} \sum_{\tau \in K'_i} v^{2\ell(\tau)} v^{-2h(w,\sigma^{\triangleleft K})} = v^{-2h(w,\sigma^{\triangleleft K})} \llbracket S \rrbracket.$$

By the construction of  $\sigma_{w,S}$  we have

$$\sum_K v^{-2h(w, \sigma^{\triangleleft K})} = v^{-2h(w, \sigma_{w,S})} \begin{bmatrix} T \\ S \end{bmatrix}. \quad (4.2.17)$$

The lemma follows by combining (4.2.16) and (4.2.17).  $\square$

**Example 4.9.** Following Example 4.7, we pick the element  $S = T \in \Gamma_T$ . Thus

$$S_A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad S_A^\dagger = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad \left( \sum_{k \leq j} (S - S^\dagger)_{ik} \right)_{ij} = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} T \\ S \end{bmatrix} = 1, \quad \llbracket S \rrbracket = \begin{bmatrix} \begin{bmatrix} & & & \\ 1 & 2 & 1 & 0 \end{bmatrix} \\ \begin{bmatrix} & & & \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{bmatrix} = [2], \quad \sigma_{1,S} = (3, 5)_c(4, 6)_c, \quad h(1, \sigma_{1,S}) = 1.$$

Also we have  $\psi_1^{-1}(S) = \{(3, 6)_c(4, 5)_c, (3, 5)_c(4, 6)_c\}$ . According to Example 3.6, we have LHS of (4.2.15)  $= v^0 + v^{-2} = v^{-2}[2] =$  RHS of (4.2.15).

For  $T \in \Theta_{B,A}$  (4.1.2) and  $S \in \Gamma_T$  (4.2.7), we set

$$n(S) = \sum_{i=1}^{r+1} \text{row}_a(S)_i, \quad (4.2.18)$$

and

$$\begin{aligned} h(T, S) &= \sum_{i=1}^{r+1} \sum_{j=-\infty}^{\infty} s_{ij} \left( \sum_{k=-\infty}^j t_{ik} - \frac{s_{ij} + 1}{2} \right) \\ &+ \sum_{i=1}^{r+1} \sum_{j=-\infty}^{\infty} (t_{1-i,-j} - s_{1-i,-j}) \left( \sum_{k=-\infty}^{j-1} t_{ik} + \sum_{k=j}^{\infty} s_{ik} - \sum_{k=j+1}^{\infty} s_{1-i,-k} \right). \end{aligned} \quad (4.2.19)$$

**Lemma 4.10.** For  $T \in \Theta_{B,A}$  and  $S \in \Gamma_T$ , we have

$$n(\sigma_{w,S}) = n(S), \quad h(T, \sigma_{w,S}) = h(T, S).$$

*Proof.* The first statement is obvious since  $n(\sigma_{w,S})$  is the number of disjoint transpositions for  $\sigma_{w,S}$ . To compute  $h(T, \sigma_{w,S})$ , We should count the elements in  $H(T, \sigma_{w,S})$ . There are  $\sum_{i=1}^{r+1} \sum_{j=-\infty}^{\infty} (t_{1-i,-j} - s_{1-i,-j}) (\sum_{k=-\infty}^{j-1} t_{ik} + \sum_{k=j}^{\infty} s_{ik} - \sum_{k=j+1}^{\infty} s_{1-i,-k})$  elements  $(u, v) \in H(T, \sigma_{w,S})$  such that  $\sigma_{w,S}(u) = u$  while there are  $\sum_{i=1}^{r+1} \sum_{j=-\infty}^{\infty} s_{ij} (\sum_{k=-\infty}^j t_{ik} - \frac{s_{ij} + 1}{2})$  elements  $(u, v) \in H(T, \sigma_{w,S})$  such that  $u$  appears in the disjoint transpositions of  $\sigma_{w,S}$ .  $\square$

**4.3. Multiplication formula for  $\mathbf{S}_{n,d}^c$ .** We are now in the position of establishing a crucial multiplication formula for affine Schur algebra  $\mathbf{S}_{n,d}^c$ . For  $A, B \in \Xi_n, T \in \Theta_{B,A}$  and  $S \in \Gamma_T$ , we recall  $\llbracket S \rrbracket, n(S), h(S, T), A^{(T-S)}$  from (4.2.13), (4.2.18), (4.2.19), (4.2.8) respectively. We further denote

$$\ell(A, B, S, T) = \ell(A) + \ell(B) - \ell(A^{(T-S)}) + \ell(w_{A,T}). \quad (4.3.1)$$

**Theorem 4.11.** *Let  $A, B \in \Xi_{n,d}$  with  $B$  tridiagonal and  $\text{row}_c(A) = \text{col}_c(B)$ . Then we have*

$$e_B e_A = \sum_{\substack{T \in \Theta_{B,A} \\ S \in \Gamma_T}} (v^2 - 1)^{n(S)} v^{2(\ell(A,B,S,T) - n(S) - h(S,T))} \llbracket A; S; T \rrbracket e_{A^{(T-S)}}, \quad (4.3.2)$$

where

$$\llbracket A; S; T \rrbracket = \frac{[A^{(T-S)}]_c!}{[T-S]![S]![A-T_\theta]_c!} \llbracket S \rrbracket. \quad (4.3.3)$$

*Proof.* We have

$$\begin{aligned} & e_B e_A \\ &= \sum_{\substack{w \in \mathcal{D}_\delta \cap W_\mu \\ \sigma \in K_w}} (v^2 - 1)^{n(\sigma)} (v^2)^{h(w,\sigma) + \ell(g_1 \sigma w g_2) - \ell(y^{(w,\sigma)})} \frac{[A^{(w,\sigma)}]_c!}{[A]_c!} e_{A^{(w,\sigma)}} \quad \text{by (2.6.9), (3.2.1)} \\ &= \sum_{\substack{w \in \mathcal{D}_\delta \cap W_\mu \\ \sigma \in K_w}} (v^2 - 1)^{n(\sigma)} (v^2)^{\ell(g_1) + \ell(w) + \ell(g_2) - \ell(y^{(w,\sigma)}) - n(\sigma) - h(w,\sigma)} \frac{[A^{(w,\sigma)}]_c!}{[A]_c!} e_{A^{(w,\sigma)}} \quad \text{by (3.2.5)} \\ &= \sum_{\substack{T \in \Theta_{B,A} \\ S \in \Gamma_T}} (v^2 - 1)^{n(S)} v^{2(\ell(A,B,S,T) - n(S) - h(S,T))} \llbracket A; S; T \rrbracket e_{A^{(T-S)}}. \quad \text{by (4.1.6), (4.2.15)} \end{aligned}$$

The proof is finished.  $\square$

In more concrete term, we can express  $\llbracket A; S; T \rrbracket$  as

$$\begin{aligned} \llbracket A; S; T \rrbracket &= \prod_{(i,j) \in I_a^+} \left[ \begin{array}{c} (A - T_\theta)_{i,j} + s_{ij} + s_{-i,-j} + \widehat{(T-S)}_{ij} + \widehat{(T-S)}_{-i,-j} \\ (A - T_\theta)_{i,j}; s_{ij}; s_{-i,-j}; \widehat{(T-S)}_{ij}; \widehat{(T-S)}_{-i,-j} \end{array} \right] \\ &\quad \cdot \prod_{k \in \{0, r+1\}} \frac{\prod_{i=1}^{s_{kk} + \widehat{(T-S)}_{kk}} [a_{kk} - 2t_{kk} - 1 + 2i]}{[s_{kk}]! [\widehat{(T-S)}_{kk}]!} \cdot \llbracket S \rrbracket. \end{aligned} \quad (4.3.4)$$

**Remark 4.12.** There is also a geometric multiplication formula in different form for  $\mathbf{S}_{n,d}^c$  in [FL16]. We have checked that the two formulas match well in various examples including the specific case in Proposition 4.13.

**4.4. Special cases of the multiplication formula.** For any matrix  $T = (t_{ij}) \in \Theta_n$ , let  $\text{diag}(T) = (\delta_{ij} t_{ij}) \in \Theta_n$  and let

$$T^\star = T - \text{diag}(T) \in \Theta_n. \quad (4.4.1)$$

We shall describe below the special case of the multiplication formula for  $\mathbf{S}_{n,d}^c$  (see Theorem 4.11) when  $B$  satisfies  $B^\star = \sum_{i=0}^r b_{i,i+1} E_\theta^{i,i+1}$  or  $B^\star = \sum_{i=0}^r b_{i+1,i} E_\theta^{i+1,i}$ . (Note that  $g_1 = \mathbb{1}$  here. A direct proof of this special case is much easier than the general case in Theorem 4.11.) The formula is analogous to the multiplication formula in affine type A in [DF14].

**Proposition 4.13.** *Let  $A, B \in \Xi_{n,d}$ . Assume  $B^\star = \sum_{i=0}^r b_{i,i+1} E_\theta^{i,i+1}$  or  $B^\star = \sum_{i=0}^r b_{i+1,i} E_\theta^{i+1,i}$  and  $\text{row}_c(A) = \text{col}_c(B)$ . Then*

$$e_B e_A = \sum_{T \in \Theta_{B,A}} v^{2(\ell(w_{A,T}) + \ell(A) - \ell(A^{(T)}))} \frac{[A^{(T)}]_c!}{[A - T^\theta]_c! [T]!} e_{A^{(T)}}.$$

*Proof.* This is a special case of (4.3.2) where  $S$  is always the zero matrix.  $\square$

Let  $\epsilon_{ij}^\theta$  be the  $(i, j)$ -th entry of  $E_\theta^{i,j}$ , that is,

$$\epsilon_{ij}^\theta = \begin{cases} 2, & \text{if } (i, j) = (0, 0) \text{ or } (r+1, r+1); \\ 1, & \text{otherwise.} \end{cases} \quad (4.4.2)$$

Below is a further specialization of Proposition 4.13 when  $B$  is a Chevalley generator. Another multiplication formula with  $B$  being a “divided power” (e.g.,  $B = \text{diag}(B) + R E_\theta^{h,h+1}$ , for  $0 \leq h \leq r$  and  $R \in \mathbb{N}$ ) can also be easily available as a specialization of Proposition 4.13, and we shall skip it. Note the formula below is analogous to the multiplication formula for Schur algebra of finite type B/C [BKLW] where a geometric approach was used. (Indeed the Hecke algebraic approach of this paper can be easily adapted to reproduce the multiplication formulas with divided powers therein.)

**Corollary 4.14.** *Let  $0 \leq h \leq r$ .*

(a) *If  $B = \text{diag}(B) + E_\theta^{h,h+1}$  and  $\text{row}_c(A) = \text{col}_c(B)$ , then*

$$e_B e_A = \sum_{\substack{p \in \mathbb{Z} \\ a_{h+1,p} \geq \epsilon_{h+1,p}^\theta}} v^{2 \sum_{j>p} a_{hj}} [a_{hp} + 1] e_{A + E_\theta^{hp} - E_\theta^{h+1,p}}.$$

(b) *If  $B = \text{diag}(B) + E_\theta^{h+1,h}$  and  $\text{row}_c(A) = \text{col}_c(B)$ , then*

$$e_B e_A = \sum_{\substack{p \in \mathbb{Z} \\ a_{hp} \geq \epsilon_{hp}^\theta}} v^{2 \sum_{j<p} a_{h+1,j}} [a_{h+1,p} + 1] e_{A - E_\theta^{hp} + E_\theta^{h+1,p}}.$$

Finally, we note the multiplication formula in Theorem 4.11 can be greatly simplified at the classical limit  $v = 1$ . Let  $(2m)!! = (2m) \cdots 4 \cdot 2$ , for  $m \geq 1$  and  $0!! = 1$ . We have the following multiplication formula for the (classical) affine Schur algebra (which is defined as for  $\mathbf{S}_{n,d}^c$  with Hecke algebra  $\mathbf{H}$  replaced by Weyl group  $W$ ).

**Corollary 4.15.** *Let  $A, B \in \Xi_{n,d}$  with  $B$  tridiagonal and  $\text{row}_c(A) = \text{col}_c(B)$ . Then we have*

$$e_B e_A|_{v=1} = \sum_{T=(t_{ij}) \in \Theta_{B,A}} \frac{\prod_{(i,j) \in I^a} (a_{ij} - t_{ij} - t_{-i,-j} + t_{i+1,j} + t_{-i+1,-j})!}{(\prod_{i=1}^n \prod_{j \in \mathbb{Z}} t_{ij}!) (\prod_{(i,j) \in I^a} (a_{ij} - t_{ij} - t_{-i,-j})!)} \cdot \prod_{i=0,r+1} \frac{(a_{ii} - 2t_{ii} + 2t_{i+1,i} - 1)!!}{(a_{ii} - 2t_{ii} - 1)!!} e_{A - T_\theta + \hat{T}_\theta}. \quad (4.4.3)$$

*Proof.* All terms in (4.3.2) having positive powers of  $(v^2 - 1)$  specialize to 0 at  $v = 1$ . Thus we only need to consider the special terms for which  $S$  is the zero matrix. Then we easily obtain (4.4.3) upon replacing all quantum numbers in (4.3.2) by ordinary numbers.  $\square$

## 5. MONOMIAL AND CANONICAL BASES FOR AFFINE SCHUR ALGEBRA

In this section, we construct canonical and monomial bases of the affine Schur algebra  $\mathbf{S}_{n,d}^c$ . The canonical basis is then identified with the canonical basis defined geometrically in [FLLLW]. The construction of the monomial basis here will play a crucial role in the setting of the stabilization algebras in the next sections. We shall show that the identification of the algebra  $\mathbf{S}_{n,d}^c$  and the geometrically defined version in [FLLLW] preserves the canonical bases.

**5.1. Bar involution on  $\mathbf{S}_{n,d}^c$ .** Recall the bar map on  $\mathbf{H}$  is an  $\mathbb{Z}$ -algebra involution  $\bar{\cdot} : \mathbf{H} \rightarrow \mathbf{H}$ , which sends  $v \mapsto v^{-1}$ ,  $T_w \mapsto T_{w^{-1}}^{-1}$ , for all  $w \in W$ . Let  $\leq$  be the (strong) Bruhat order on  $W$ . Following [KL79], denote by  $\{C'_w\}$  the Kazhdan-Lusztig basis of the Hecke algebra  $\mathbf{H}$  characterized by Conditions (C1)–(C2) below:

- (C1)  $C'_w$  is bar-invariant;
- (C2)  $C'_w = v^{-\ell(w)} \sum_{y \leq w} P_{yw} T_y$ , where  $P_{ww} = 1$  and  $P_{yw} \in \mathbb{Z}[v^2]$  for  $y < w$  has degree in  $v \leq \ell(w) - \ell(y) - 1$ .

For  $\lambda, \mu \in \Lambda$  (2.2.1), set  $g_{\lambda\mu}^+$  to be the *longest* element in  $W_\lambda g W_\mu$  for  $g \in \mathcal{D}_{\lambda\mu}$ , and set  $w_\circ^\mu = 1_{\mu\mu}^+$  to be the longest element in the (finite) parabolic subgroup  $W_\mu = W_\mu 1 W_\mu$ . Recall  $x_\mu$  for  $\mu \in \Lambda$  from (2.3.1).

**Lemma 5.1.** *Let  $\lambda, \mu \in \Lambda$ ,  $g \in \mathcal{D}_{\lambda\mu}$ , and  $\delta = \delta(\lambda, g, \mu)$ ; see Proposition 2.4. Then we have*

- (a)  $g_{\lambda\mu}^+ = w_\circ^\lambda g w_\circ^\mu$ , and  $\ell(g_{\lambda\mu}^+) = \ell(w_\circ^\lambda) + \ell(g) - \ell(w_\circ^\delta) + \ell(w_\circ^\mu)$ .
- (b)  $W_\lambda g W_\mu = \{w \in W \mid g \leq w \leq g_{\lambda\mu}^+\}$ .
- (c)  $T_{W_\lambda g W_\mu} = v^{\ell(g_{\lambda\mu}^+)} C'_{g_{\lambda\mu}^+} + \sum_{\substack{y \in \mathcal{D}_{\lambda\mu} \\ y < g}} c_{y,g}^{(\lambda,\mu)} C'_{y_{\lambda\mu}^+}$ , for  $c_{y,g}^{(\lambda,\mu)} \in \mathcal{A}$ . Moreover,  $x_\mu = v^{\ell(w_\circ^\mu)} C'_{w_\circ^\mu}$ .

*Proof.* See [Cur85, Theorem 1.2, (1.11)] and [DDPW, Corollary 4.19].  $\square$

Following [Du92, Proposition 3.2], we define a bar involution  $\bar{\cdot}$  on  $\mathbf{S}_{n,d}^c$  as follows: for each  $f \in \text{Hom}_{\mathbf{H}}(x_\mu \mathbf{H}, x_\lambda \mathbf{H})$ , let  $\bar{f} \in \text{Hom}_{\mathbf{H}}(x_\mu \mathbf{H}, x_\lambda \mathbf{H})$  be the  $\mathbf{H}$ -linear map which sends  $C'_{w_\circ^\mu}$  to  $\overline{f(C'_{w_\circ^\mu})}$ , that is,

$$\bar{f}(x_\mu H) = v^{2\ell(w_\circ^\mu)} \overline{f(x_\mu)} H, \quad \text{for } H \in \mathbf{H}.$$

For  $A = \kappa(\lambda, g, \mu) \in \Xi_n$ , we have  $e_A(x_\mu) = \phi_{\lambda\mu}^g(x_\mu) = T_{W_\lambda g W_\mu}$ ; see (2.3.2) and (2.5.3). Hence by Lemma 5.1(c) we have

$$e_A(C'_{w_\circ^\mu}) = v^{\ell(g_{\lambda\mu}^+) - \ell(w_\circ^\mu)} C'_{g_{\lambda\mu}^+} + \sum_{\substack{y \in \mathcal{D}_{\lambda\mu} \\ y < g}} v^{-\ell(w_\circ^\mu)} c_{y,g}^{(\lambda,\mu)} C'_{y_{\lambda\mu}^+}, \quad (5.1.1)$$

$$\overline{e_A(C'_{w_\circ^\mu})} = v^{\ell(w_\circ^\mu) - \ell(g_{\lambda\mu}^+)} C'_{g_{\lambda\mu}^+} + \sum_{\substack{y \in \mathcal{D}_{\lambda\mu} \\ y < g}} v^{\ell(w_\circ^\mu)} \overline{c_{y,g}^{(\lambda,\mu)}} C'_{y_{\lambda\mu}^+}. \quad (5.1.2)$$

**5.2. A standard basis in  $\mathbf{S}_{n,d}^c$ .** For each  $A = \kappa(\lambda, g, \mu) \in \Xi_{n,d}$ , we set

$$\ell(A) = \ell(g),$$

the length of the corresponding Weyl group element  $g$ . By rephrasing Lemma 2.1, we obtain the following expression of  $\ell(A)$  as a polynomial in the entries of  $A$ .



**Proposition 5.2.** *Let  $A = (a_{ij}) \in \Xi_n$  and recall  $a'_{ij}$  in (2.3.10). Then we have*

$$\ell(A) = \frac{1}{2} \left( \sum_{(i,j) \in I^+} \left( \sum_{\substack{x < i \\ y > j}} + \sum_{\substack{x > i \\ y < j}} \right) a'_{ij} a_{xy} \right). \quad (5.2.1)$$

We define

$$d_A = \frac{1}{2} \left( \sum_{(i,j) \in I^+} \left( \sum_{x \leq i, y > j} + \sum_{x \geq i, y < j} \right) a'_{ij} a_{xy} \right), \quad \text{for } A \in \Xi_{n,d}. \quad (5.2.2)$$

We shall see in Proposition 5.3(1) that  $d_A \in \mathbb{N}$ . Set

$$[A] = v^{-d_A} e_A. \quad (5.2.3)$$

Then it follows by Lemma 2.5 that  $\{[A] \mid A \in \Xi_{n,d}\}$  forms an  $\mathcal{A}$ -basis of  $\mathbf{S}_{n,d}^c$ , which we call the *standard basis*.

Note that it was defined in [FLLW] that

$$d_A = \frac{1}{2} \left( \sum_{\substack{i \geq k, j < l \\ i \in [0, n-1]}} a_{ij} a_{kl} - \sum_{i \geq 0 > j} a_{ij} - \sum_{i \geq r+1 > j} a_{ij} \right), \quad \text{for } A \in \Xi_{n,d}. \quad (5.2.4)$$

As shown in [FLLW, (4.1.1), Lemma 4.1.1],  $d_A$  (5.2.4) is the dimension of a *generalized* affine Schubert variety  $X_A^L$ . One can show the two different definitions of  $d_A$  in (5.2.2) and (5.2.4) coincide (we leave the proof to the reader).

**Proposition 5.3.** *Let  $A = \kappa(\lambda, g, \mu) \in \Xi_{n,d}$ . Then we have*

- (1)  $d_A = \ell(g_{\lambda\mu}^+) - \ell(w_\circ^\mu)$ ;
- (2)  $\overline{[A]} \in [A] + \sum_{\substack{y \in \mathcal{D}_{\lambda\mu} \\ y < g}} \mathcal{A}[\kappa(\lambda, y, \mu)]$ .

*Proof.* Set  $\delta = \delta(A) = (\delta_0, \delta_1, \dots, \delta_{r+1})$ ; see (2.5.4). Recall  $\ell(A) = \ell(g)$  and (5.2.1). Hence by applying Lemma 5.1(a) and then Proposition 5.2 we have

$$\begin{aligned} \ell(g_{\lambda\mu}^+) - \ell(w_\circ^\mu) - \ell(g) &= \ell(w_\circ^\lambda) - \ell(w_\circ^\delta) \\ &= \lambda_0^2 + \sum_{i=1}^r \binom{\lambda_i}{2} + \lambda_{r+1}^2 - \left( (\delta_0)^2 + \sum_{i=1}^{r'} \binom{\delta_i}{2} + (\delta_{r'+1})^2 \right) \\ &= 2 \sum_{0 \leq j < x} a'_{0j} a_{0x} + \sum_{j > 0} \binom{a_{0j} + 1}{2} + \sum_{\substack{j < y \\ 1 \leq i \leq r}} a_{ij} a_{iy} \\ &\quad + 2 \sum_{j < x \leq r+1} a_{r+1,j} a'_{r+1,x} + \sum_{j < r+1} \binom{a_{r+1,j} + 1}{2} \\ &= \frac{1}{2} \left( \sum_{(i,j) \in I^+} \left( \sum_{\substack{x=i \\ y>j}} + \sum_{\substack{x=i \\ y<j}} \right) a'_{ij} a_{xy} \right) \\ &= d_A - \ell(A) = d_A - \ell(g). \end{aligned}$$

Hence we have  $d_A = \ell(g_{\lambda\mu}^+) - \ell(w_\circ^\mu)$ , proving (1).

Equations (5.1.1)–(5.1.2) can be rewritten as

$$[A](C'_{w_o^\mu}) = C'_{g_{\lambda\mu}^+} + \sum_{\substack{y \in \mathcal{D}_{\lambda\mu} \\ y < g}} v^{-\ell(g_{\lambda\mu}^+)} c_{y,g}^{(\lambda,\mu)} C'_{y_{\lambda\mu}^+}, \quad (5.2.5)$$

$$\overline{[A]}(C'_{w_o^\mu}) = C'_{g_{\lambda\mu}^+} + \sum_{\substack{y \in \mathcal{D}_{\lambda\mu} \\ y < g}} v^{\ell(g_{\lambda\mu}^+)} \overline{c_{y,g}^{(\lambda,\mu)}} C'_{y_{\lambda\mu}^+}. \quad (5.2.6)$$

Inverting Equation (5.2.5), we see that  $C'_{g_{\lambda\mu}^+} \in [A](C'_{w_o^\mu}) + \sum_{\substack{y \in \mathcal{D}_{\lambda\mu} \\ y < g}} v^{-1} \mathcal{A}[\kappa(\lambda, y, \mu)](C'_{w_o^\mu})$ . Now Part (2) follows by from this and (5.2.6).  $\square$

For  $A \in \Xi_n$ , we let

$$\sigma_{ij}(A) = \sum_{x \leq i, y \geq j} a_{xy}. \quad (5.2.7)$$

Now we define a partial order  $\leq_{\text{alg}}$  on  $\Xi_n$  by letting, for  $A, B \in \Xi_n$ ,

$$A \leq_{\text{alg}} B \Leftrightarrow \text{row}_c(A) = \text{row}_c(B), \text{col}_c(A) = \text{col}_c(B), \text{ and } \sigma_{ij}(A) \leq \sigma_{ij}(B), \forall i < j. \quad (5.2.8)$$

We denote  $A <_{\text{alg}} B$  if  $A \leq_{\text{alg}} B$  and  $A \neq B$ . In the following the expression “lower terms” represents a linear combination of smaller elements with respect to  $\leq_{\text{alg}}$ .

**Lemma 5.4.** *Assume that  $A = \kappa(\lambda, g, \mu)$  and  $B = \kappa(\lambda, h, \mu)$ . If  $h \leq g$  then  $B \leq_{\text{alg}} A$ .*

*Proof.* By [BB05, Theorem 8.4.8], the Bruhat ordering  $h \leq g$  is equivalent to that  $h[s, t] \leq g[s, t]$  for all  $s, t \in \mathbb{Z}$ , where  $g[s, t] = |\{a \in \mathbb{Z} \mid a \leq s, g(a) \geq t\}|$ . Thanks to the bijections  $R_x^\lambda \cap gR_y^\mu \leftrightarrow \{a \in R_y^\mu \mid g(a) \in R_x^\lambda\}$  for  $x, y \in \mathbb{Z}$ , we have, for  $i < j$ ,

$$\sigma_{ij}(A) = \sum_{x \leq i, y \geq j} a_{xy} = \sum_{x \geq -i, y \leq -j} |R_x^\lambda \cap gR_y^\mu| = g[s, t],$$

where  $s$  is the largest element in  $R_{-i}^\lambda$  and  $t$  is the smallest element in  $R_{-j}^\mu$ . Therefore,  $\sigma_{ij}(B) = h[s, t] \leq g[s, t] = \sigma_{ij}(A)$ , for all  $s, t$ , and so  $B \leq_{\text{alg}} A$ .  $\square$

**Corollary 5.5.** *For any  $A \in \Xi_{n,d}$ , we have  $\overline{[A]} = [A] + \text{lower terms}$ .*

*Proof.* It follows by combining Proposition 5.3 and Lemma 5.4.  $\square$

**5.3. Multiplication formula using  $[A]$ .** Let us reformulate the multiplication formula for  $\mathbf{S}_{n,d}^c$  (Theorem 4.11) in terms of the standard basis.

**Theorem 5.6.** *Let  $A, B \in \Xi_{n,d}$  with  $B$  tridiagonal and  $\text{row}_c(A) = \text{col}_c(B)$ . Then we have*

$$[B][A] = \sum_{\substack{T \in \Theta_{B,A} \\ S \in \Gamma_T}} v^{\beta(A,S,T)} (v^2 - 1)^{n(S)} \overline{[A; S; T]} [A^{(T-S)}], \quad (5.3.1)$$

where  $\beta(A, S, T)$  is given in (5.3.4) below.

The explicit formula for  $\beta(A, S, T)$  can be derived as follows. Let  $\gamma(A, S, T)$  be the integer such that

$$\overline{[A; S; T]} = v^{\gamma(A,S,T)} [A; S; T]. \quad (5.3.2)$$

**Lemma 5.7.** *Let  $A, B \in \Xi_{n,d}$  with  $B$  tridiagonal. Let  $T \in \Theta_{B,A}$  and  $S \in \Gamma_T$ . Then*

$$\begin{aligned} \gamma(A, S, T) = & - \sum_{(i,j) \in I_a^+} \left( s_{\theta,ij} + (\widehat{T-S})_{\theta,ij} \right) \left( s_{\theta,ij} + (\widehat{T-S})_{\theta,ij} + 2a_{ij} - 2t_{\theta,ij} - 1 \right) \\ & - \sum_{k \in \{0, r+1\}} \left( a_{kk} - 1 - (T-S)_{\theta,kk} + (\widehat{T-S})_{\theta,kk} \right) \left( s_{\theta,kk} + (\widehat{T-S})_{\theta,kk} \right) \\ & + 2 \sum_{i=1}^n \sum_{j \in \mathbb{Z}} \left( \binom{(T-S)}{2} + \binom{s_{ij}}{2} \right) \\ & + 2 \sum_{i=1}^{r+1} \sum_{j \in \mathbb{Z}} s_{i,j+1}^\dagger (s_{i,j+1}^\dagger - \sum_{k \leq j} (S - S^\dagger)_{ik}) - \binom{s_{i,j+1}^\dagger}{2}. \end{aligned} \quad (5.3.3)$$

In particular, by setting  $d'(A) = 2\ell(A) - d_A$ , we have

$$\beta(A, S, T) = d'(B) + d'(A) - d'(A^{(T-S)}) + 2\ell(w_{A,T}) - 2n(S) - 2h(S, T) + \gamma(A, S, T). \quad (5.3.4)$$

*Proof.* By a direct computation using (2.6.3), we have

$$\begin{aligned} \overline{[A]}^\dagger &= v^{-2 \sum_{i=1}^n \sum_{j \in \mathbb{Z}} \binom{a_{ij}}{2}} [A]^\dagger, \\ \overline{[A]}^\dagger_{\mathbf{c}} &= v^{-2(a'_{00})^2 - 2(a'_{r+1, r+1})^2 - 2 \sum_{(i,j) \in I_a^+} \binom{a_{ij}}{2}} [A]^\dagger_{\mathbf{c}}, \\ \overline{[S]} &= v^{2 \sum_{i=1}^{r+1} \sum_{j \in \mathbb{Z}} s_{i,j+1}^\dagger (s_{i,j+1}^\dagger - \sum_{k \leq j} (S - S^\dagger)_{ik}) - 2 \binom{s_{i,j+1}^\dagger}{2}} \llbracket S \rrbracket. \end{aligned}$$

The lemma follows by putting them together.  $\square$

**5.4. The canonical basis for  $\mathbf{S}_{n,d}^c$ .** Recall  $C'_{w_0^\mu}$  and  $C'_{g_{\lambda\mu}^+}$  from Section 5.1. For any  $A = \kappa(\lambda, g, \mu) \in \Xi_{n,d}$ , following [Du92] we define

$$\{A\} \in \text{Hom}_{\mathbf{H}}(x_\mu \mathbf{H}, x_\lambda \mathbf{H}), \text{ and hence } \{A\} \in \mathbf{S}_{n,d}^c,$$

by requiring

$$\{A\}(C'_{w_0^\mu}) = C'_{g_{\lambda\mu}^+}. \quad (5.4.1)$$

By the definition of  $\overline{\phantom{x}}$ , we have  $\overline{\{A\}}(C'_{w_0^\mu}) = C'_{g_{\lambda\mu}^+}$ , and hence  $\overline{\{A\}} = \{A\}$ .

Following [Du92, (2.c), Lemma 3.8], we have

$$\{A\} = [A] + \sum_{y < g} v^{\ell(y_{\lambda\mu}^+) - \ell(g_{\lambda\mu}^+)} P_{y_{\lambda\mu}^+, g_{\lambda\mu}^+} [\kappa(\lambda, y, \mu)].$$

Recalling (C2) in §5.1, we have

$$\{A\} \in [A] + \sum_{y < g} v^{-1} \mathbb{Z}[v^{-1}] [\kappa(\lambda, y, \mu)]. \quad (5.4.2)$$

By (5.2.5),  $\{\{A\} \mid A \in \Xi_{n,d}\}$  forms a basis of  $\mathbf{S}_{n,d}^c$ , which is called *the canonical basis*. We summarize this as follows.

**Theorem 5.8.** *There exists a canonical basis  $\mathfrak{B}_{n,d}^c := \{\{A\} \mid A \in \Xi_{n,d}\}$  for  $\mathbf{S}_{n,d}^c$ , which is characterized by the bar-invariance and the property (5.4.2).*

Recall from [FLLLW] that there is a bar involution on  $\mathbf{S}_{n,d}^{\mathfrak{c},\text{geo}}$  and a (geometric) partial ordering  $\leq_{\text{geo}}$  on  $\Xi_{n,d}$ . It was shown *loc. cit.* that there exists a canonical basis of  $\mathbf{S}_{n,d}^{\mathfrak{c},\text{geo}}$ ,  $\{\{A\}^{\text{geo}} \mid A \in \Xi_{n,d}\}$  (note  $\{A\}^{\text{geo}}$  was denoted as  $\{A\}_d$  in [FLLLW, §4.2]). By definition,  $\{A\}^{\text{geo}}$  is bar invariant, and  $\{A\}^{\text{geo}} \in [A]^{\text{geo}} + \sum_{B <_{\text{geo}} A} v^{-1}\mathbb{Z}[v^{-1}][B]^{\text{geo}}$ . Recall the algebra isomorphism  $\psi : \mathbf{S}_{n,d}^{\mathfrak{c},\text{geo}} \rightarrow \mathbf{S}_{n,d}^{\mathfrak{c}}$  from Proposition 2.14.

**Proposition 5.9.** *The isomorphism  $\psi : \mathbf{S}_{n,d}^{\mathfrak{c},\text{geo}} \rightarrow \mathbf{S}_{n,d}^{\mathfrak{c}}$  commutes with the bar maps and preserves the canonical bases; that is,  $\psi$  sends  $\{A\}^{\text{geo}}$  to  $\{A\}$  for  $A \in \Xi_{n,d}$ .*

*Proof.* The bar involution on  $\mathbf{S}_{n,d}^{\mathfrak{c}}$  is simply an algebraic reformulation of the bar involution on  $\mathbf{S}_{n,d}^{\mathfrak{c},\text{geo}}$  in [FLLLW] (which goes back to [BLM90]; see [Du92]). Hence the bar operations on  $\mathbf{S}_{n,d}^{\mathfrak{c}}$  and  $\mathbf{S}_{n,d}^{\mathfrak{c},\text{geo}}$  are compatible, i.e.,  $\bar{\phantom{x}} \circ \psi = \psi \circ \bar{\phantom{x}}$ .

We note here a crucial fact that  $A \leq_{\text{geo}} B$  implies  $A \leq_{\text{alg}} B$  (cf. [FLLLW, §4.2], where  $\leq_{\text{alg}}$  is denoted by  $\leq$ ; the counterparts in finite type A and in the affine type A can be found in [BLM90, Lu99]). Hence  $\psi(\{A\}^{\text{geo}})$  is bar invariant and  $\psi(\{A\}^{\text{geo}}) \in [A] + \sum_{B <_{\text{alg}} A} v^{-1}\mathbb{Z}[v^{-1}][B]$ . Hence  $\psi(\{A\}^{\text{geo}}) = \{A\}$  by the characterization of  $\{A\}$ .  $\square$

**5.5. A leading term.** A pair  $(B, A)$  of matrices in  $\Xi_n \times \Xi_n$  is called *admissible* if the following conditions (a)–(b) hold:

- (a)  $B^\star = \sum_{i=1}^n b_{i,i+1} E_\theta^{i,i+1}$  (see (4.4.1) for notation  $B^\star$ );
- (b)  $A^\star = \sum_{j=1}^x \sum_{i=1}^n a_{i,i+j} E_\theta^{i,i+j}$  for some  $x \in \mathbb{N}$ , where  $a_{i,i+x} \geq b_{i,i+1}$  for all  $i$ . (If  $a_{i,i+x} \neq 0$  for some  $i$ , we say that  $A$  is of depth  $x$ .)

Assume that  $A, B \in \Xi_{n,d}$  with  $B$  tridiagonal and  $\text{row}_\mathfrak{c}(A) = \text{col}_\mathfrak{c}(B)$ . We produce a matrix

$$M(B, A) \in \Xi_{n,d} \quad (5.5.1)$$

as follows. For each row  $i$ , find the unique  $j$  such that  $b_{i,i+1} \in (\sum_{y>j} a_{iy} \dots \sum_{y \geq j} a_{iy}]$ , and denote

$$\mathbf{T} = \mathbf{T}_{B,A} = \sum_{i=1}^n \left( (b_{i,i+1} - \sum_{y>j} a_{iy}) E_\theta^{ij} + \sum_{y>j} a_{iy} E_\theta^{iy} \right) \in \Theta_n. \quad (5.5.2)$$

Set  $M(B, A) = A^{(\mathbf{T})}$ ; see (4.2.8).

**Lemma 5.10.** *Assume that  $A, B \in \Xi_{n,d}$  with  $B$  tridiagonal and  $\text{row}_\mathfrak{c}(A) = \text{col}_\mathfrak{c}(B)$ . The highest term (with respect to  $\leq_{\text{alg}}$ ) on the RHS of (5.3.1) exists and is equal to  $[M(B, A)]$  (with coefficient being some nonzero scalar in  $\mathcal{A}$ ).*

*Proof.* Note that

$$\sigma_{ij}(A^{(E^{xy})}) = \begin{cases} \sigma_{ij}(A) + 1 & \text{if } j < i = x - 1, j \leq y, \\ \sigma_{ij}(A) - 1 & \text{if } j > i = x, j \geq y, \\ \sigma_{ij}(A) & \text{otherwise.} \end{cases} \quad (5.5.3)$$

It follows that  $A^{(E^{ij})} <_{\text{alg}} A^{(E^{i,j+1})}$  for all  $i, j \in \mathbb{Z}$ . Therefore, for any  $T \in \Theta_{B,A}$ ,  $S \in \Gamma_T$  we have  $A^{(T-S)} \leq_{\text{alg}} A^{(T)} \leq_{\text{alg}} A^{(\mathbf{T})} = M(B, A)$ .  $\square$

The corollary below is a direct consequence of (5.5.3).

**Corollary 5.11.** *Let  $A', A, B \in \Xi_{n,d}$ , with  $B$  tridiagonal. If  $A' <_{\text{alg}} A$ , then  $M(B, A') <_{\text{alg}} M(B, A)$ .*

Recall  $\llbracket A; S; T \rrbracket$  from (4.3.3).

**Lemma 5.12.** *Let  $A, B \in \Xi_n$ . If  $(B, A)$  is admissible, then  $\llbracket A; 0; T_{B,A} \rrbracket = 1$ .*

*Proof.* Write  $T = T_{B,A}$ , cf. (5.5.2). As  $(B, A)$  is admissible, we have  $T = \sum_{i=1}^n b_{i,i+1} E_{\theta}^{i,i+1}$ . Moreover, we have

$$\llbracket A; 0; T \rrbracket = \frac{1}{[T]!} \frac{[A + \hat{T}_{\theta}^+ - T_{\theta}]!_{\mathbf{c}}}{[A - T_{\theta}]!_{\mathbf{c}}} = \frac{1}{\prod_{i=1}^n [m_i]} \cdot \prod_{i=1}^n [m_i] = 1.$$

The lemma is proved.  $\square$

**Lemma 5.13.** *Let  $A, B \in \Xi_{n,d}$  with  $B$  tridiagonal and  $\text{row}_{\mathbf{c}}(A) = \text{col}_{\mathbf{c}}(B)$ . If  $B' <_{\text{alg}} B$ , then  $B'$  is also tridiagonal. Moreover, we have  $M(B', A) <_{\text{alg}} M(B, A)$ .*

*Proof.* Since  $B' \leq_{\text{alg}} B$ , we have

$$\sigma_{i,i+2}(B') \leq \sigma_{i,i+2}(B) = 0 \quad \text{for } i = 1, \dots, n.$$

Therefore  $\sigma_{i,i+2}(B') = 0$  for all  $i$  and hence  $B'$  is tridiagonal. Also, we have

$$\sigma_{i,i+1}(B') = b'_{i,i+1} \leq \sigma_{i,i+1}(B) = b_{i,i+1} \quad \text{for } i = 1, \dots, n.$$

It follows from this and the definition (5.5.1) that  $M(B', A) <_{\text{alg}} M(B, A)$ .  $\square$

By Lemma 5.12, we can rewrite (5.3.1) as

$$[B][A] = v^{\beta(A,0,T)}[M(B, A)] + \text{lower terms.} \quad (5.5.4)$$

One can show that  $\beta(A, 0, T) = 0$  by a direct but lengthy computation. Here we present a formal argument of this fact via the bar involution.

**Lemma 5.14.** *If  $(B, A)$  is admissible (and we retain the notation in Conditions (a)–(b) in §5.5), then  $\beta(A, 0, T) = 0$ . Hence, we have*

$$[B][A] = [M(B, A)] + \text{lower terms.}$$

Moreover, we have  $M(B, A) = A - \sum_{i=1}^n b_{i,i+1}(E_{\theta}^{i,i+1} - E_{\theta}^{i-1,i+1})$ .

*Proof.* Write  $M = M(B, A)$ , cf. (5.5.1). By taking bar on both sides of (5.5.4), we obtain

$$\overline{[B]} \overline{[A]} = v^{-\beta(A,0,T)} \overline{[M]} + \text{lower terms.} \quad (5.5.5)$$

By Proposition 5.3, we can write

$$\overline{[B]} = [B] + \sum_{B' <_{\text{alg}} B} \gamma_{B,B'} [B'], \quad \overline{[A]} = [A] + \sum_{A' <_{\text{alg}} A} \gamma_{A,A'} [A']. \quad (5.5.6)$$

For any  $B' <_{\text{alg}} B$ , by Lemma 5.13 and Corollary 5.11 we have

$$M(B', A') <_{\text{alg}} M(B', A) <_{\text{alg}} M.$$

Therefore, by (5.5.6) and (5.5.4) we have

$$\overline{[B]} \overline{[A]} = [B][A] + \text{lower terms} = v^{\beta(A,0,T)}[M] + \text{lower terms.} \quad (5.5.7)$$

By comparing the leading coefficients in (5.5.5) and (5.5.7), we obtain  $\beta(A, 0, T) = 0$ .

The formula for  $M(B, A)$  follows by definition of (5.5.1).  $\square$

**5.6. A semi-monomial basis.** Below we provide an algorithm that constructs a monomial basis in a diagonal-by-diagonal manner involving only admissible pairs. A similar algorithm for monomial basis in affine type A was given in [LL15].

**Algorithm 5.15.** For each  $A = (a_{ij}) \in \Xi_{n,d}$ , we construct tridiagonal matrices  $A^{(1)}, \dots, A^{(x)} \in \Xi_{n,d}$  as follows:

- (1) Initialization: set  $t = 0$ , and set  $C^{(0)} = A$ .
- (2) If  $C^{(t)} \in \Xi_{n,d}$  is a tridiagonal matrix, then set  $x = t + 1$ ,  $A^{(x)} = C^{(t)}$ , and the algorithm terminates.

Otherwise, for  $C^{(t)} = (c_{ij}^{(t)}) \in \Xi_{n,d}$ , set  $k = \max \{|i - j| \mid c_{ij}^{(t)} \neq 0\} \geq 2$ , and

$$T^{(t)} = \sum_{i=1}^n c_{i,i+k}^{(t)} E^{i,i+k}.$$

- (3) Define matrices

$$A^{(t+1)} = \sum_{i=1}^n c_{i,i+k}^{(t)} E_{\theta}^{i,i+1} + \text{a diagonal determined by (5.6.1)},$$

$$C^{(t+1)} = C^{(t)} - T_{\theta}^{(t)} + \widetilde{T}_{\theta}^{(t)},$$

where  $\widetilde{X}$  is the matrix obtained from  $X$  by shifting every entry down by one row; also see (2.3.7) for notation.

- (4) Increase  $t$  by one and then go to Step (2).

Here the diagonal entries of  $A^{(t)}$  are uniquely determined by

$$\text{row}_a(A^{(1)}) = \text{row}_a(A), \quad \text{row}_a(A^{(t+1)}) = \text{col}_a(A^{(t)}), \quad 1 \leq t \leq x - 1. \quad (5.6.1)$$

**Theorem 5.16.**

- (1) For each  $A \in \Xi_{n,d}$ , the tridiagonal matrices  $A^{(1)}, \dots, A^{(x)} \in \Xi_{n,d}$  in Algorithm 5.15 satisfy that

$$m'_A := [A^{(1)}][A^{(2)}] \cdots [A^{(x)}] = [A] + \text{lower terms}. \quad (5.6.2)$$

- (2) The set  $\{m'_A \mid A \in \Xi_{n,d}\}$  forms an  $\mathcal{A}$ -basis of  $\mathbf{S}_{n,d}^c$  (called a semi-monomial basis).
- (3) The algebra  $\mathbf{S}_{n,d}^c$  is generated by the  $[A]$ , for tridiagonal matrices  $A \in \Xi_{n,d}$ .

*Proof.* Algorithm 5.15 guarantees that each pair  $(A^{(j)}, C^{(j)})$  is admissible, and moreover,  $M(A^{(j)}, C^{(j)}) = C^{(j-1)}$  for  $2 \leq j \leq x$ . By construction we have  $A^{(x)} = C^{(x-1)}$ . Hence by Corollary 5.11 and by Lemma 5.14, we have

$$\begin{aligned} [A^{(1)}][A^{(2)}] \cdots ([A^{(x-1)}][A^{(x)}]) &= [A^{(1)}][A^{(2)}] \cdots ([A^{(x-1)}][C^{(x-1)}]) \\ &= [A^{(1)}][A^{(2)}] \cdots [A^{(x-2)}]([C^{(x-2)}] + \text{lower terms}) \\ &= [A] + \text{lower terms}. \end{aligned}$$

This proves (1). Part (2) follows from (1) as the transition matrix between the standard basis  $\{[A] \mid A \in \Xi_{n,d}\}$  and  $\{m'_A \mid A \in \Xi_{n,d}\}$  is uni-lower-triangular. Part (3) follows from (2). The theorem is proved.  $\square$

In general, the element  $m'_A$  is not bar-invariant since for an arbitrary tridiagonal matrix  $B$ ,  $[B]$  is not necessarily bar-invariant.

**5.7. A monomial basis for  $\mathbf{S}_{n,d}^c$ .** Recall from (5.5.1) that  $M(B, A)$  is defined for admissible pairs. We now extend the definition to arbitrary pairs in  $\Xi_n \times \Xi_n$  using the semi-monomial bases. Let  $B^{(1)}, \dots, B^{(x)} \in \Xi_n$  be the tridiagonal matrices in Algorithm 5.15 and Theorem 5.16 (now associated to  $B$ ) satisfying

$$[B^{(1)}][B^{(2)}] \cdots [B^{(x)}] = [B] + \text{lower terms.} \quad (5.7.1)$$

We define matrices

$$M^{(x+1)} = A, \quad M^{(i)} = M(B^{(i)}, M^{(i+1)}) \quad \text{for } 1 \leq i \leq x, \quad (5.7.2)$$

and then set

$$M(B, A) = M^{(1)}. \quad (5.7.3)$$

We have the following generalization of Corollary 5.11 and Lemma 5.13.

**Lemma 5.17.** *Let  $A', A, B, B' \in \Xi_{n,d}$ .*

- (a) *If  $A' <_{\text{alg}} A$ , then  $M(B, A') <_{\text{alg}} M(B, A)$ .*
- (b) *If  $B' <_{\text{alg}} B$ , then  $M(B', A) <_{\text{alg}} M(B, A)$ .*

*Proof.* Part (a) follows from applying Corollary 5.11 repeatedly.

(b) Assume now  $B' <_{\text{alg}} B$ . We keep the notations in (5.7.1), (5.7.2) and (5.7.3). Again, Algorithm 5.15 produces tridiagonal matrices  $B'^{(j)}$  for  $1 \leq j \leq y$  satisfying

$$[B'^{(1)}][B'^{(2)}] \cdots [B'^{(y)}] = [B'] + \text{lower terms.} \quad (5.7.4)$$

It follows from the construction of semi-monomial basis that  $y \leq x$ , and we define matrices  $B'^{(i)}$  for  $y < i \leq x$  to be the diagonal matrix with  $\text{col}_c(B'^{(i)}) = \text{row}_c(A)$ . Therefore, we have  $B'^{(i)} \leq_{\text{alg}} B^{(i)}$  for all  $i$ , and the strict inequality holds for some  $i_0 \geq 1$ . Following (5.7.2), we define

$$M'^{(x+1)} = A, \quad M'^{(i)} = M(B'^{(i)}, M'^{(i+1)}) \quad \text{for } 1 \leq i \leq x.$$

It follows by definition (5.7.3) that  $M(B', A) = M'^{(1)}$ . We claim that

$$M'^{(i)} \leq_{\text{alg}} M^{(i)}, \quad \forall 1 \leq i \leq x+1; \text{ moreover the inequality is strict for } 1 \leq i \leq i_0. \quad (5.7.5)$$

We prove (5.7.5) by a downward induction on  $i$ . The base case of the induction is trivial as  $M'^{(x+1)} = A \leq_{\text{alg}} A = M^{(x+1)}$ . Assume  $M'^{(i+1)} \leq_{\text{alg}} M^{(i+1)}$  for some  $i \geq 1$ . Then by using first Lemma 5.13 and then Corollary 5.11 we have

$$M'^{(i)} = M(B'^{(i)}, M'^{(i+1)}) \leq_{\text{alg}} M(B'^{(i)}, M^{(i+1)}) \leq_{\text{alg}} M(B^{(i)}, M^{(i+1)}) = M^{(i)}. \quad (5.7.6)$$

This proves the first statement in (5.7.5). Now for  $i = i_0$ , the first inequality in (5.7.6) must be strict by Lemma 5.13, and for  $i < i_0$ , the second inequality in (5.7.6) must be strict by Corollary 5.11. The proof of (5.7.5) is completed.

The statement (5.7.5) for  $i = 1$  gives us  $M(B', A) <_{\text{alg}} M(B, A)$ , whence (b).  $\square$

Modifying (5.6.2), for  $A \in \Xi_n$ , we define

$$m_A = \{A^{(1)}\}\{A^{(2)}\} \cdots \{A^{(x)}\}. \quad (5.7.7)$$

**Theorem 5.18.** *The element  $m_A$ , for  $A \in \Xi_{n,d}$ , is bar-invariant, and moreover,  $m_A = [A] + \text{lower terms}$ . Hence, the set  $\{m_A \mid A \in \Xi_{n,d}\}$  forms an  $\mathcal{A}$ -basis of  $\mathbf{S}_{n,d}^c$ .*

*Proof.* By definition  $m_A$  is bar invariant. It follows by Corollary 5.11 and Lemma 5.13 that  $m_A = [A] + \text{lower terms}$ . Hence, the set  $\{m_A \mid A \in \Xi_{n,d}\}$  forms an  $\mathcal{A}$ -basis of  $\mathbf{S}_{n,d}^c$  since  $\{[A] \mid A \in \Xi_{n,d}\}$  is a basis.  $\square$

**Remark 5.19.** Traditionally, the monomial basis was introduced as an intermediate step toward construction of canonical basis. We have reversed the order of introducing these two bases for  $\mathbf{S}_{n,d}^c$ . Algorithm 5.15, which plays a fundamental role in constructing the monomial basis, will be adapted to construct the monomial and canonical bases for the stabilization algebra arising from the family (or its variants) of Schur algebras  $\mathbf{S}_{n,d}^c$  as  $d$  varies in the next sections.

## 6. STABILIZATION ALGEBRA $\dot{\mathbf{K}}_n^c$ ARISING FROM AFFINE SCHUR ALGEBRAS

In this section, we shall establish a stabilization property for the family of affine Schur algebras  $\mathbf{S}_{n,d}^c$  as  $d$  varies, which leads to a quantum algebra  $\dot{\mathbf{K}}_n^c$ . The multiplication formula for  $\mathbf{S}_{n,d}^c$  with tridiagonal generators in the previous section plays a fundamental role. We construct a monomial basis and a stably canonical basis for  $\dot{\mathbf{K}}_n^c$ .

### 6.1. A BLM-type stabilization. Let

$$\begin{aligned} \tilde{\Xi}_n = \left\{ A = (a_{ij}) \in \text{Mat}_{\mathbb{Z} \times \mathbb{Z}}(\mathbb{Z}) \mid a_{-i,-j} = a_{ij} = a_{i+n,j+n}, \forall i, j \in \mathbb{Z}; \right. \\ \left. a_{k\ell} \geq 0, \forall k \neq \ell \in \mathbb{Z}; a_{00}, a_{r+1,r+1} \text{ are odd} \right\}. \end{aligned} \quad (6.1.1)$$

Extending the partial ordering  $\leq_{\text{alg}}$  for  $\Xi_n$ , we define a partial ordering  $\leq_{\text{alg}}$  on  $\tilde{\Xi}_n$  using the same recipe (5.2.8).

For each  $A \in \tilde{\Xi}_n$  and  $p \in \mathbb{N}$ , let  ${}_p A = A + pI$  where  $I$  is the identity matrix. Then  ${}_p A \in \Xi_n$  for even  $p \gg 0$ . Let  $v', v''$  be two indeterminates (independent of  $v$ ), and  $\mathcal{R}_1$  be the subring of  $\mathbb{Q}(v)[v']$  generated by

$$\prod_{i=1}^t \frac{v^{2(a+i)} v'^2 - 1}{v^{2i} - 1}, \quad \prod_{i=1}^t \frac{v^{4(a+i)} v'^2 - 1}{v^{2i} - 1}, \quad \text{and } v^a, \quad \text{for } a \in \mathbb{Z}, t \in \mathbb{Z}_{>0}. \quad (6.1.2)$$

Let  $\mathcal{R}_2$  be the subring of  $\mathbb{Q}(v)[v', v'^{-1}]$  generated by

$$\begin{aligned} \prod_{i=1}^t \frac{v^{2(a+i)} v'^2 - 1}{v^{2i} - 1}, \quad \prod_{i=1}^t \frac{v^{4(a+i)} v'^2 - 1}{v^{2i} - 1}, \\ \prod_{i=1}^t \frac{v^{-2(a+i)} v'^{-2} - 1}{v^{-2i} - 1}, \quad \prod_{i=1}^t \frac{v^{-4(a+i)} v'^{-2} - 1}{v^{-2i} - 1}, \quad \text{and } v^a, \quad \text{for } a \in \mathbb{Z}, t \in \mathbb{Z}_{>0}. \end{aligned} \quad (6.1.3)$$

Let  $\mathcal{R}_3 = \mathcal{R}_2[v'', v''^{-1}]$  be a subring of  $\mathbb{Q}(v)[v', v'^{-1}, v'', v''^{-1}]$ .

**Proposition 6.1.** *Let  $A_1, \dots, A_f \in \tilde{\Xi}_n$  be such that  $\text{col}_c(A_i) = \text{row}_c(A_{i+1})$  for all  $i$ . Then there exists matrices  $Z_1, \dots, Z_m \in \tilde{\Xi}_n$  and  $G_i(v, v') \in \mathcal{R}_2$  such that for even integer  $p \gg 0$ ,*

$$[{}_p A_1][{}_p A_2] \cdots [{}_p A_f] = \sum_{i=1}^m G_i(v, v^{-p})[{}_p Z_i]. \quad (6.1.4)$$



*Proof.* The proof follows the basic idea as for [BLM90, 4.2], except one step which is more complicated in our setting (because of the complexity of the multiplication formula in Theorem 5.6); this is also responsible for the subtle difference of presence of  $\mathcal{R}_2$  instead of  $\mathcal{R}_1$  in the proposition. We will present this step in more detail.

Similar to [BLM90], for  $A \in \tilde{\Xi}_n$ , we introduce

$$\Psi(A) = \sum_{(i,j) \in I^+} \binom{|i-j|+1}{2} a_{ij} \in \mathbb{N}. \quad (6.1.5)$$

Noting  $\Psi(A) = \sum_{(i,j) \in I^+} \sigma_{ij}(A)$  (see (5.2.7) for notation  $\sigma_{ij}(A)$ ), we have

$$\Psi(A) < \Psi(B) \text{ if } A <_{\text{alg}} B, \text{ for } A, B \in \tilde{\Xi}_n.$$

Assume for now we have proved the identity (6.1.4) when  $f = 2$  and  $A_1$  is tridiagonal. By iteration inducting on  $f$ , (6.1.4) holds for general  $f$  when  $A_i$  for all  $1 \leq i \leq f$  are tridiagonal. Exactly as in proof of [BLM90, 4.2], using this special case we just proved together with Theorem 5.16 we can prove (6.1.4) when  $f = 2$  (and for general  $A_1$ ) by induction on  $\Psi(A_1)$ . Then by iteration inducting on  $f$ , the identity (6.1.4) in full generality follows.

It remains to verify the identity (6.1.4) when  $f = 2$  and  $B = A_1$  is tridiagonal. Set  $A = A_2$ . This is the step more complicated than [BLM90, 4.2] which we alluded to at the beginning. For even integer  $p \gg 0$  such that all entries in  $A_i$  are in  $\mathbb{N}$ , we can apply Theorem 5.6 and obtain

$$[{}_p B][{}_p A] = \sum_{\substack{T \in \Theta_{B, {}_p A} \\ S \in \Gamma_T}} v^{\beta({}_p A, S, T)} (v^2 - 1)^{n(S)} \overline{[{}_p A; S; T]} [{}_p A^{(T-S)}]. \quad (6.1.6)$$

Recall  $\beta(A, S, T)$  from (5.3.4), and hence

$$\beta({}_p A, S, T) = d'({}_p B) + d'({}_p A) - d'({}_p A^{(T_\theta - S_\theta)}) + 2\ell(w_{{}_p A, T}) + \gamma({}_p A, S, T) - 2n(S) - 2h(S, T).$$

We compute the difference  $\beta({}_p A, S, T) - \beta(A, S, T)$  term by term as follows:

$$\begin{aligned} d'({}_p B) - d'(B) &= -p \sum_{i=0}^{2r+1} b_{i,i+1}, \\ d'({}_p A) - d'(A) &= \frac{p}{2} \sum_{i=0}^{2r+1} \left( \sum_{x < i, y < i} a_{xy} - \sum_{y > i} a_{iy} \right) \\ &\quad + \frac{p}{2} \sum_{(i,j) \in I^+} \left( \sum_{x=1}^r \left( \sum_{i < x < j} + \sum_{i > x > j} \right) a_{ij} - \sum_{i \neq j} a_{ij} \right), \\ \ell(w_{{}_p A, T}) - \ell(w_{A, T}) &= p \sum_{i=0}^{2r+1} \sum_{j > i} t_{ij}, \\ \gamma({}_p A, S, T) - \gamma(A, S, T) &= -p \left( \sum_{k \in \{0, r+1\}} (S + \hat{T} - \hat{S})_{\theta, kk} - 2 \sum_{(i,j) \in I_{\mathfrak{a}}^+} (S + \hat{T} - \hat{S})_{\theta, ij} \right). \end{aligned} \quad (6.1.7)$$

Combining these gives us

$$\beta({}_p A, S, T) = \beta(A, S, T) + pG_1(B, A, S, T),$$

where  $G_1(B, A, S, T)$  depends only on the entries of  $B, A, S, T$  (and is independent of  $p$ ).

On the other hand, set

$$\begin{aligned}
a_{ij}^{(1)} &= p\delta_{ij} + (A - T_\theta)_{ij} + s_{-i,-j} + (\widehat{T - S})_{ij} + (\widehat{T - S})_{-i,-j}, \\
a_{ij}^{(2)} &= p\delta_{ij} + (A - T_\theta)_{ij} + (\widehat{T - S})_{ij} + (\widehat{T - S})_{-i,-j}, \\
a_{ij}^{(3)} &= p\delta_{ij} + (A - T_\theta)_{ij} + (\widehat{T - S})_{-i,-j}, \\
a_{ij}^{(4)} &= p\delta_{ij} + (A - T_\theta)_{ij}, \\
a_{kk}^{(5)} &= p + a_{kk} - 2t_{kk} - 1 + 2s_{kk} \in 2\mathbb{Z}, \\
a_{kk}^{(6)} &= p + a_{kk} - 2t_{kk} - 1 \in 2\mathbb{Z}.
\end{aligned} \tag{6.1.8}$$

We have

$$\begin{aligned}
\llbracket_p A; S; T \rrbracket &= \prod_{(i,j) \in I_a^+} \left( \prod_{l=1}^{s_{ij}} \frac{[a_{ij}^{(1)} + l]}{[l]} \prod_{l=1}^{s_{-i,-j}} \frac{[a_{ij}^{(2)} + l]}{[l]} \prod_{l=1}^{(\widehat{T-S})_{ij}} \frac{[a_{ij}^{(3)} + l]}{[l]} \prod_{l=1}^{(\widehat{T-S})_{-i,-j}} \frac{[a_{ij}^{(4)} + l]}{[l]} \right) \\
&\quad \cdot \prod_{k \in \{0, r+1\}} \left( \prod_{l=1}^{(\widehat{T-S})_{kk}} \frac{[a_{kk}^{(5)} + 2l]}{[l]} \prod_{l=1}^{s_{kk}} \frac{[a_{kk}^{(6)} + 2l]}{[l]} \right) \cdot \llbracket S \rrbracket.
\end{aligned}$$

Hence  $v^{\beta(pA, S, T)} \overline{\llbracket_p A; S; T \rrbracket}$  is of the form  $G(v, v^{-p})$  for some  $G(v, v') \in \mathcal{R}_2$ .

This finishes the proof of the identity (6.1.4) when  $f = 2$  and  $B = A_1$  is tridiagonal. The proposition is proved.  $\square$

## 6.2. Stabilization of bar involutions.

**Proposition 6.2.** *For any  $A \in \tilde{\Xi}_n$ , there exist matrices  $T_1, \dots, T_s \in \tilde{\Xi}_n$  and  $H_i(v, v', v'') \in \mathcal{R}_3$  such that, for even integer  $p \gg 0$ ,*

$$\overline{[pA]} = \sum_{i=1}^s H_i(v, v^{-p}, v^{p^2}) [pT_i]. \tag{6.2.1}$$

*Proof.* We follow the strategy of the proof of [BLM90, 4.3] closely. We will be sketchy on the almost identical steps, except one extra step which we will present the detail below. We can assume without loss of generality that  $A \in \Xi_n$ , by replacing  $A$  by  $p_0 A$  if necessary.

We prove the identity (6.2.1) by induction on  $\Psi(A)$ . If  $\Psi(A) = 0$ , then  $A$  is diagonal, and  $\overline{[pA]} = [pA]$ . Assume  $\Psi(A) \geq 1$ . Combining Theorem 5.16 and Proposition 6.1, for  $p \gg 0$  we have

$$[pA] = [pA_1] \cdots [pA_f] + \sum_k G_k(v, v^{-p}, v^{p^2}) [pZ_k], \tag{6.2.2}$$

where  $A_j$ 's are all tridiagonal,  $G_k \in \mathcal{R}_3$  and  $\Psi(Z_k) < \Psi(A)$ . By inductive hypothesis, there are  $T''_{i,k} \in \tilde{\Xi}_n$  and  $H''_{i,k} \in \mathcal{R}_3$  such that  $\overline{[pZ_k]} = \sum_i H''_{i,k}(v, v^{-p}, v^{p^2}) [pT''_{i,k}]$ .

Assume for now the identity (6.2.1) holds for all tridiagonal  $A$ . Hence, there are  $T'_{i,j} \in \tilde{\Xi}_n$  and  $H'_{i,j} \in \mathcal{R}_3$  such that  $\overline{[pA_j]} = \sum_i H'_{i,j}(v, v^{-p}, v^{p^2}) [pT'_{i,j}]$ . Applying the bar map to both

sides of (6.2.2), we have

$$\begin{aligned} \overline{[{}_p A]} &= \left( \sum_i H'_{i,1}(v, v^{-p}, v^{p^2}) [{}_p T'_{i,1}] \right) \cdots \left( \sum_i H'_{i,f}(v, v^{-p}, v^{p^2}) [{}_p T'_{i,f}] \right) \\ &\quad + \sum_{i,k} \overline{G}_k(v, v^{-p}, v^{p^2}) H''_{i,k}(v, v^{-p}, v^{p^2}) [{}_p T''_{i,k}]. \end{aligned}$$

Now the identity (6.2.1) in full generality follows from this by Proposition 6.1.

It remains to prove the identity (6.2.1) when  $A$  is tridiagonal, which is the extra step we alluded to above. (In the setting of [BLM90, 4.3], this step is trivial where  $A$  is bidiagonal, and  $\overline{[{}_p A]} = [{}_p A]$ .)

For  $\lambda \in \Lambda_{r,d}$  we define  ${}_p \lambda \in \Lambda_{r,d+p(r+1)}$  by

$${}_p \lambda = \left( \lambda_0 + \frac{p}{2}, \lambda_1 + p, \dots, \lambda_r + p, \lambda_{r+1} + \frac{p}{2} \right).$$

We fix a tridiagonal matrix  $A = \kappa(\lambda, g, \mu) \in \Xi_n$  for some  $\lambda, \mu \in \Lambda_{r,d}$ ,  $g \in \mathcal{D}_{\lambda\mu} \subseteq W(d)$ . (Here we have denoted the affine Weyl group  $W$  as  $W(d)$  to indicate its dependence on  $d$ ; we shall see  ${}_p g \in W(d + p(r+1))$  below.)

Recall from (3.1.3)  $g = \prod_{i=0}^r g^{(i)}$ , where  $g^{(i)} \in \text{Stab}(R_{3i+1}^{\delta(A)} \cup R_{3i+2}^{\delta(A)})$  has a reduced expression

$$g^{(i)} = (s_{m_i+\beta_i} \cdots s_{m_i+2} s_{m_i+1}) (s_{m_i+\beta_i+1} \cdots s_{m_i+2}) \cdots (s_{m_i+\beta_i+\alpha_i-1} \cdots s_{m_i+\alpha_i}),$$

where  $\alpha_i \geq 0, \beta_i \geq 0$  are such that  $R_{3i-2}^{\delta(A)} = (m_i \dots m_i + \alpha_i]$  and  $R_{3i-1}^{\delta(A)} = (m_i + \alpha_i \dots m_i + \alpha_i + \beta_i]$ . Hence  ${}_p A = \kappa({}_p \lambda, {}_p g, {}_p \mu)$  where  ${}_p g \in W(d + p(r+1))$  is uniquely determined by letting, for  $1 \leq x \leq d + p(r+1)$ ,

$${}_p g(x) = \begin{cases} x + a_{i-1,i} & \text{if } x \in R_{3i+1}^{\delta({}_p A)}, \\ x - a_{i,i-1} & \text{if } x \in R_{3i+2}^{\delta({}_p A)}, \\ x & \text{otherwise.} \end{cases}$$

Equivalently, by setting  $p_i = p(\frac{1}{2} + i)$ , we have  ${}_p g = \prod_{i=0}^r {}_p g^{(i)}$  where

$${}_p g^{(i)} = (s_{p_i+m_i+\beta_i} \cdots s_{p_i+m_i+2} s_{p_i+m_i+1}) \cdots (s_{p_i+m_i+\beta_i+\alpha_i-1} \cdots s_{p_i+m_i+\alpha_i}).$$

Now we define  ${}_p x \in W(d + p(r+1))$  for any  $x \in W(d)$  with  $x \leq g$  as follows. Let  $x = \prod_{i=0}^r x^{(i)}$ , and  $x^{(i)} = s_{j_{i,1}} \cdots s_{j_{i,k_i}}$  be a reduced expression, which is a subexpression of  $g^{(i)}$ . Define

$${}_p x = \prod_{i=0}^r {}_p x^{(i)} \quad \text{where} \quad {}_p x^{(i)} = s_{p_i+j_{i,1}} \cdots s_{p_i+j_{i,k_i}}.$$

In other words,  ${}_p x$  is obtained from  $x$  by enlarging the domain (i.e.,  $R_{3i}^{\delta(A)}$  is replaced by  $R_{3i}^{\delta({}_p A)}$ ) on which it acts as an identity map. The non-trivial action of  ${}_p x$  is the same as the non-trivial action of  $x$  (up to a shift by  $p_i$ ). We observe that

$$\{w \in \mathcal{D}_{p\lambda, p\mu} \mid w \leq {}_p g\} = \{{}_p x \in \mathcal{D}_{\lambda\mu} \mid x \leq g\}. \quad (6.2.3)$$

Indeed, for  $w \leq {}_p g$ , it follows from construction that  $x$  is a subexpression of  ${}_p g$  and hence there is a unique  $x \leq g$  such that  ${}_p x = w$  obtained by subtracting  $p_i$  from the indices of simple reflections. It follows by Lemma 2.3 that  $x \in \mathcal{D}_{\lambda\mu}$ . The other inclusion in (6.2.3) is similar.

As the non-trivial portions of  $({}_p x, {}_p g)$  are those of  $(x, g)$ , the corresponding Kazhdan-Lusztig polynomials coincide, i.e.,

$$P_{x,g} = P_{{}_p x, {}_p g}. \quad (6.2.4)$$

Denote by  ${}_p g_{\lambda\mu}^+$  the longest element in  $(W_{p\lambda})_p g(W_{p\mu})$ . By Lemma 5.1(c), we have

$$T_{(W_{p\lambda})_p g(W_{p\mu})} = v^{\ell({}_p g_{\lambda\mu}^+)} C'_{{}_p g_{\lambda\mu}^+} + \sum_{\substack{w \in \mathcal{D}_{p\lambda, p\mu} \\ w < {}_p g}} c_{w, {}_p g}^{(p\lambda, p\mu)} C'_{w_{p\lambda, p\mu}^+}. \quad (6.2.5)$$

By (6.2.3), Equation (6.2.5) can be written as

$$T_{(W_{p\lambda})_p g(W_{p\mu})} = v^{\ell({}_p g_{\lambda\mu}^+)} C'_{{}_p g_{\lambda\mu}^+} + \sum_{\substack{x \in \mathcal{D}_{\lambda\mu} \\ x < g}} c_{p x, {}_p g}^{(p\lambda, p\mu)} C'_{p x_{\lambda\mu}^+}. \quad (6.2.6)$$

In particular,  $T_{(W_{p\mu})_1(W_{p\mu})} = x_{p\mu} = v^{\ell(w_{\circ}^{p\mu})} C'_{w_{\circ}^{p\mu}}$ , where

$$\begin{aligned} \ell(w_{\circ}^{p\mu}) &= \ell(w_{\circ}^{\mu}) + \left(\mu_0 + \frac{p}{2}\right)^2 - \mu_0^2 + \sum_{i=1}^r \left( \binom{\mu_i + p}{2} - \binom{\mu_i}{2} \right) + \left(\mu_{r+1} + \frac{p}{2}\right)^2 - \mu_{r+1}^2 \\ &= \ell(w_{\circ}^{\mu}) + p\mu_0 + \frac{p^2}{4} + \sum_{i=1}^r p\mu_i + r \binom{p}{2} + p\mu_{r+1} + \frac{p^2}{4} \\ &= \ell(w_{\circ}^{\mu}) + p(d - \frac{r}{2}) + p^2(\frac{r+1}{2}). \end{aligned} \quad (6.2.7)$$

By Lemma 5.1 again, for any  $x \in \mathcal{D}_{\lambda\mu}$  such that  $x \leq g$  with  $A_x = \kappa(\lambda, x, \mu)$ , we have

$$\begin{aligned} &\ell({}_p x_{\lambda\mu}^+) - \ell(x_{\lambda\mu}^+) \\ &= \left( \ell(w_{\circ}^{p\lambda}) - \ell(w_{\circ}^{\lambda}) \right) + \ell({}_p x) - \ell(g) - \left( \ell(w_{\circ}^{\delta({}_p A_x)}) - \ell(w_{\circ}^{\delta(A)}) \right) + \left( \ell(w_{\circ}^{p\mu}) - \ell(w_{\circ}^{\mu}) \right) \\ &= p(d + |A_x^*| - \frac{r}{2}) + p^2(\frac{r+1}{2}). \end{aligned} \quad (6.2.8)$$

Here  $|A_x^*|$  is the sum of off-diagonal entries of  $A_x$  over  $I_{\mathfrak{a}}^+$ . Hence  $v^{\ell({}_p x_{\lambda\mu}^+)}$  is a specialization at  $(v, v^{-p}, v^{p^2})$  of some element in  $\mathcal{R}_3$ , for  $x \leq g, x \in \mathcal{D}_{\lambda\mu}$ . By [Cur85, Theorem 1.10, Corollary 1.12], the matrix  $(c_{p x, {}_p g}^{(p\lambda, p\mu)})_{x, g \in \mathcal{D}_{\lambda\mu}}$  is the inverse of the matrix  $(v^{-\ell({}_p g_{\lambda\mu}^+)} P_{{}_p x_{\lambda\mu}^+, {}_p g_{\lambda\mu}^+})_{x, g}$ . Hence it follows by (6.2.4) and (6.2.8) that  $c_{p x, {}_p g}^{(p\lambda, p\mu)}$  is a specialization at  $(v, v^{-p}, v^{p^2})$  of some element in  $\mathcal{R}_3$ .

Therefore we deduce from (6.2.6) that

$$\left( \overline{[{}_p A]} - [{}_p A] \right) (C'_{w_{\circ}^{p\mu}}) = \sum_{\substack{x \in \mathcal{D}_{\lambda\mu} \\ x < g}} \left( v^{\ell({}_p g_{\lambda\mu}^+)} \overline{c_{p x, {}_p g}^{(p\lambda, p\mu)}} - v^{-\ell({}_p g_{\lambda\mu}^+)} c_{p x, {}_p g}^{(p\lambda, p\mu)} \right) C'_{p x_{\lambda\mu}^+}, \quad (6.2.9)$$

where the coefficients on the RHS are specializations at  $(v, v^{-p}, v^{p^2})$  of some elements in  $\mathcal{R}_3$ . By an induction on the Bruhat order, there exist  $H_i \in \mathcal{R}_3$  and  $w^{(i)} \in \mathcal{D}_{\lambda\mu}$  satisfying

$$\overline{[{}_p A]} = [{}_p A] + \sum_i H_i(v, v^{-p}, v^{p^2}) [{}_p T_i], \quad (6.2.10)$$

where  $T_i = \kappa(\lambda, w^{(i)}, \mu)$ . This finishes the proof of the identity (6.2.1) when  $A$  is tridiagonal.

The proposition is proved.  $\square$

Let  $\dot{\mathbf{K}}_n^c$  be the free  $\mathbb{Q}(v)$ -module with basis given by the symbols  $[A]$  for  $A \in \tilde{\Xi}_n$  (which will be called a standard basis of  $\dot{\mathbf{K}}_n^c$ ). By Propositions 6.1–6.2 and applying a specialization at  $v' = 1$ , we have the following corollary.

**Corollary 6.3.** *There is a unique associative  $\mathbb{Q}(v)$ -algebra structure on  $\dot{\mathbf{K}}_n^c$  with multiplication given by*

$$[A_1] \cdot [A_2] \cdot \dots \cdot [A_f] = \begin{cases} \sum_{i=1}^m G_i(v, 1)[Z_i] & \text{if } \text{col}_c(A_i) = \text{row}_c(A_{i+1}) \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the map  $^- : \dot{\mathbf{K}}_n^c \rightarrow \dot{\mathbf{K}}_n^c$  given by  $\overline{v^k[A]} = v^{-k} \sum_{i=1}^s H_i(v, 1, 1)[T_i]$  is a  $\mathbb{Q}$ -linear involution.

**6.3. Multiplication formula for  $\dot{\mathbf{K}}_n^c$ .** The following multiplication formula in  $\dot{\mathbf{K}}_n^c$  follows directly from Theorem 5.6 by the stabilization construction (see Proposition 6.1).

**Theorem 6.4.** *Let  $A, B \in \tilde{\Xi}_n$  with  $B$  tridiagonal and  $\text{row}_c(A) = \text{col}_c(B)$ . Then we have*

$$[B][A] = \sum_{\substack{T \in \tilde{\Theta}_{B,A} \\ S \in \Gamma_T}} v^{\beta(A,S,T)} (v^2 - 1)^{n(S)} \overline{[A; S; T]} [A^{(T-S)}], \quad (6.3.1)$$

where

$$\tilde{\Theta}_{B,A} = \{T \in \Theta_n \mid t_{ij} + t_{-i,-j} \leq a_{ij} \text{ unless } i = j, \text{ row}_a(T)_i = b_{i-1,i} \text{ for all } i\}. \quad (6.3.2)$$

**6.4. Monomial and stably canonical bases for  $\dot{\mathbf{K}}_n^c$ .** Recall an admissible pair of matrices  $(A, B) \in \Xi_n \times \Xi_n$  is defined by Conditions (a)–(b) in §5.5. We extend this to a definition of admissible pair  $(A, B) \in \tilde{\Xi}_n \times \tilde{\Xi}_n$  by imposing the same conditions.

The following lemma is similar to Lemma 5.14 and can be proved by the same arguments.

**Lemma 6.5.** *If  $(B, A)$  is admissible and  $A$  is of depth  $x$  (see (a)–(b) in §5.5), then*

$$[B][A] = [M(B, A)] + \text{lower terms},$$

where  $M(B, A) = A - \sum_{i=1}^n b_{i,i+1} (E_\theta^{i,i+x} - E_\theta^{i-1,i+x})$ .

The following is an analogue of Theorem 5.16.

**Proposition 6.6.** *For any  $A \in \tilde{\Xi}_n$  of depth  $x$ , there exist tridiagonal matrices  $A^{(1)}, \dots, A^{(x)}$  in  $\tilde{\Xi}_n$  satisfying  $\text{row}_c(A^{(1)}) = \text{row}_c(A)$ ,  $\text{col}_c(A^{(x)}) = \text{col}_c(A)$ ,  $\text{col}_c(A^{(i)}) = \text{row}_c(A^{(i+1)})$  for  $1 \leq i \leq x-1$  and  $A^{(i)} - (\sum_{j=1}^n \sum_{m \leq j-x+i} a_{m,j+1}) E_\theta^{j,j+1}$  is diagonal for all  $1 \leq i \leq x$  such that*

$$m'_A := [A^{(1)}][A^{(2)}] \cdots [A^{(x)}] = [A] + \text{lower terms} \in \dot{\mathbf{K}}_n^c. \quad (6.4.1)$$

*Proof.* Lemma 6.5 leads to an algorithm (almost) identical to Algorithm 5.15, which produces the tridiagonal matrices  $A^{(i)}$  (whose diagonal entries might be negative) as needed. The rest of the proof is the same as for Theorem 5.16.  $\square$

Hence  $\{m'_A \mid A \in \tilde{\Xi}_n\}$  forms a basis for  $\dot{\mathbf{K}}_n^c$  (called a *semi-monomial basis*).

Let  $B \in \tilde{\Xi}_n$  be tridiagonal. The analogue of Lemma 5.13 holds here, and so we have

$$\overline{[B]} \in [B] + \sum_{B' \text{ tridiagonal}, B' <_{\text{alg}} B} \mathcal{A}[B'].$$

If  $B$  is diagonal, set  $\{B\} = [B]$ . Arguing inductively on the partial order  $\leq_{\text{alg}}$  and using a standard argument (cf. [Lu93, 24.2.1]) there exists a unique element  $\{B\} \in \dot{\mathbf{K}}_n^c$  such that

$$\overline{\{B\}} = \{B\}, \quad \{B\} \in [B] + \sum_{B' \text{ tridiagonal}, B' <_{\text{alg}} B} v^{-1} \mathbb{Z}[v^{-1}][B'].$$

Modifying (6.4.1), for  $A \in \tilde{\Xi}_n$ , we now define

$$m_A = \{A^{(1)}\} \{A^{(2)}\} \cdots \{A^{(x)}\}. \quad (6.4.2)$$

**Theorem 6.7.**

- (a) We have  $m_A \in [A] + \sum_{B \in \tilde{\Xi}_n, B <_{\text{alg}} A} \mathcal{A}[B]$ , for any  $A \in \tilde{\Xi}_n$ . Hence the set  $\{m_A \mid A \in \tilde{\Xi}_n\}$  forms an  $\mathcal{A}$ -basis of  $\dot{\mathbf{K}}_n^c$  (called a monomial basis of  $\dot{\mathbf{K}}_n^c$ ).
- (b) There exists a unique basis  $\mathfrak{B}^c = \{\{A\} \mid A \in \tilde{\Xi}_n\}$  of  $\dot{\mathbf{K}}_n^c$  such that  $\overline{\{A\}} = \{A\}$  and  $\{A\} \in [A] + \sum_{B \in \tilde{\Xi}_n, B <_{\text{alg}} A} v^{-1} \mathbb{Z}[v^{-1}][B]$  ( $\mathfrak{B}^c$  is called stably canonical basis of  $\dot{\mathbf{K}}_n^c$ ).

*Proof.* With the help of Proposition 6.6, the assertion for monomial basis is proved in the same way as for Theorem 5.18, and hence skipped.

It follows by (a) that  $\overline{[A]} = [A] + \text{lower terms} \in \dot{\mathbf{K}}_n^c$ . The canonical basis follows by this and a standard argument (cf. [Lu93, 24.2.1]).  $\square$

**6.5. Isomorphism  $\dot{\mathbf{K}}_n^{c, \text{geo}} \cong \dot{\mathbf{K}}_n^c$ .** Recall from [FLLW, Section 9.4], there is an associative  $\mathbb{Q}(v)$ -algebra  $\dot{\mathbf{K}}_n^{c, \text{geo}}$ , with a standard basis given by  $\{[A]^{\text{geo}} \mid A \in \tilde{\Xi}_n\}$ , and a bar map defined by a stabilization procedure. (Note  $\dot{\mathbf{K}}_n^{c, \text{geo}}$  and  $[A]^{\text{geo}}$  was denoted by  $\dot{\mathbf{K}}_n^c$  and  $[A]$  therein.)

**Proposition 6.8.** *There is an algebra isomorphism  $\dot{\mathbf{K}}_n^{c, \text{geo}} \xrightarrow{\cong} \dot{\mathbf{K}}_n^c$ ,  $[A]^{\text{geo}} \mapsto [A]$ . The isomorphism commutes with the bar maps, and it preserves the canonical bases, i.e.,  $\{A\}^{\text{geo}} \mapsto \{A\}$  for all  $A$ .*

*Proof.* Recall from Proposition 2.14 the algebra isomorphisms  $\psi : \mathbf{S}_{n,d}^{c, \text{geo}} \xrightarrow{\cong} \mathbf{S}_{n,d}^c$ , for all  $d$ . As the algebra structures  $\dot{\mathbf{K}}_n^{c, \text{geo}}$  and  $\dot{\mathbf{K}}_n^c$  arise from the same stabilization procedure from the family of algebras  $\{\mathbf{S}_{n,d}^{c, \text{geo}}\}_{d \geq 0}$  and  $\{\mathbf{S}_{n,d}^c\}_{d \geq 0}$ , respectively (by comparing Proposition 6.1 with [FLLW, Proposition 9.2.5] and noting  $\mathcal{R}_1 \subset \mathcal{R}_2$ ), we obtain the algebra isomorphism  $\dot{\mathbf{K}}_n^{c, \text{geo}} \simeq \dot{\mathbf{K}}_n^c$ .

By Proposition 5.9,  $\psi : \mathbf{S}_{n,d}^{c, \text{geo}} \rightarrow \mathbf{S}_{n,d}^c$  commutes with the bar maps. As the bar maps on  $\dot{\mathbf{K}}_n^{c, \text{geo}}$  and  $\dot{\mathbf{K}}_n^c$  are defined by the same stabilization procedure from the bar maps on the family of algebras  $\{\mathbf{S}_{n,d}^{c, \text{geo}}\}_{d \geq 0}$  and  $\{\mathbf{S}_{n,d}^c\}_{d \geq 0}$ , respectively (by comparing Proposition 6.2 with [FLLW, Proposition 9.2.7] and noting  $\mathcal{R}_2 \subset \mathcal{R}_3$ ), the compatibility of the 2 bar maps under the isomorphism follows. The last claim is proved by the same argument as for Schur algebras in Proposition 5.9.  $\square$

**Remark 6.9.** The algebra isomorphism  $\dot{\mathbf{K}}_n^{c, \text{geo}} \cong \dot{\mathbf{K}}_n^c$  in Proposition 6.8 allows us to transport further results in [FLLW, §9.7] over here, and so we do not give new proofs here. Set  $\mathbb{Q}\mathbf{S}_{n,d}^c = \mathbb{Q}(v) \otimes_{\mathbb{Z}[v, v^{-1}]} \mathbf{S}_{n,d}^c$ . In particular, the map

$$\Psi_d : \dot{\mathbf{K}}_n^c \longrightarrow \mathbb{Q}\mathbf{S}_{n,d}^c, \quad [A] \mapsto \begin{cases} [A], & \text{for } A \in \Xi_{n,d}, \\ 0, & \text{for } A \in \tilde{\Xi}_n \setminus \Xi_{n,d} \end{cases}$$

is a homomorphism which preserves the canonical bases; that is,  $\Psi_d$  sends  $\{A\} \mapsto \{A\}$  for  $A \in \Xi_{n,d}$  and  $\{A\} \mapsto 0$  otherwise.

## 7. THE QUANTUM SYMMETRIC PAIR $(\mathbf{K}_n, \mathbf{K}_n^c)$

In this section, we construct an algebra  $\mathbf{K}_n^c$  as a subalgebra of a completion of the algebra  $\hat{\mathbf{K}}_n^c$ , after reviewing a similar type A construction. We study a comultiplication on  $\mathbf{K}_n^c$ , show that  $\mathbf{K}_n^c$  is a coideal subalgebra of  $\mathbf{K}_n$  (a stabilization algebra of affine type A), and that  $(\mathbf{K}_n, \mathbf{K}_n^c)$  forms a quantum symmetric pair.

**7.1. The algebra  $\mathbf{K}_n$  of type A.** In this subsection we review briefly the affine type A construction (which goes back in finite type A to [BLM90]).

Recall that

$$\mathbb{Z}_n = \{\lambda = (\lambda_i)_{i \in \mathbb{Z}} \mid \lambda_i \in \mathbb{Z}, \lambda_i = \lambda_{i+n}, \forall i \in \mathbb{Z}\}. \quad (7.1.1)$$

Let  $\hat{\mathbf{K}}_n$  be the  $\mathbb{Q}(v)$ -vector space of all (possibly infinite) linear combinations  $\sum_{A \in \tilde{\Theta}_n} \xi_A {}^a[A]$ , for  $\xi_A \in \mathbb{Q}(v)$  and  ${}^a[A] \in \hat{\mathbf{K}}_n$ , such that the sets  $\{A \in \tilde{\Theta}_n \mid \xi_A \neq 0, \text{row}_a(A) = \lambda\}$  and  $\{A \in \tilde{\Theta}_n \mid \xi_A \neq 0, \text{col}_a(A) = \lambda\}$  are finite, for any  $\lambda \in \mathbb{Z}_n$ .

The following multiplication defines an algebra structure for  $\hat{\mathbf{K}}_n$ :

$$\left( \sum_A \xi_A {}^a[A] \right) \left( \sum_B \eta_B {}^a[B] \right) = \sum_{A,B} \xi_A \eta_B ({}^a[A] {}^a[B]).$$

Let

$$\Theta_n^0 = \{A = (a_{ij}) \in \Theta_n \mid a_{ii} = 0, \forall i \in \mathbb{Z}\}.$$

For  $\alpha \in \mathbb{Z}_n$ , we set

$$D_\alpha = \text{diag}(\alpha) \in \Theta_n.$$

For each  $\mathbf{j} \in \mathbb{Z}_n$  and  $A \in \Theta_n^0$ , we set

$${}^a A(\mathbf{j}) = \sum_{\alpha \in \mathbb{Z}_n} v^{\mathbf{j} \cdot \alpha} {}^a [A + D_\alpha] \in \hat{\mathbf{K}}_n, \quad (7.1.2)$$

where  $\mathbf{j} \cdot \alpha = \sum_{i=1}^n j_i \alpha_i$ . For  $t \in \mathbb{Z}$  we define

$$\vec{t} = (\dots, t, t, t, \dots) \in \mathbb{Z}^{\mathbb{Z}}. \quad (7.1.3)$$

In particular, we have

$${}^a A(\vec{0}) = \sum_{\alpha \in \mathbb{Z}_n} {}^a [A + D_\alpha], \quad {}^a 0(\mathbf{j}) = \sum_{\alpha \in \mathbb{Z}_n} v^{\mathbf{j} \cdot \alpha} {}^a [D_\alpha]. \quad (7.1.4)$$

Let  $\mathbf{K}_n$  be the  $\mathbb{Q}(v)$ -subspace of  $\hat{\mathbf{K}}_n$  spanned by  $\{{}^a A(\mathbf{j}) \mid A \in \Theta_n^0, \mathbf{j} \in \mathbb{Z}_n\}$ . Let

$$\Theta_n^1 = \{A = (a_{ij}) \in \Theta_n^0 \mid a_{ij} = 0 \text{ unless } j - i = 1, \forall i, j \in \mathbb{Z}\},$$

$$\Theta_n^{-1} = \{A = (a_{ij}) \in \Theta_n^0 \mid a_{ij} = 0 \text{ unless } j - i = -1, \forall i, j \in \mathbb{Z}\}, \quad \Theta_n^{\pm 1} = \Theta_n^1 \cup \Theta_n^{-1}.$$

The proposition below is an affine type A analogue of a result in [BLM90].

**Proposition 7.1** ([DF15, Theorem 1.1]). *The space  $\mathbf{K}_n$  is a subalgebra of  $\hat{\mathbf{K}}_n$  generated by  ${}^a A(\vec{0})$ ,  ${}^a 0(\mathbf{j})$ , for  $A \in \Theta_n^{\pm 1}$  and  $\mathbf{j} \in \mathbb{Z}_n$ . Moreover, the algebra  $\mathbf{K}_n$  is isomorphic to the quantum affine  $\mathfrak{gl}_n$ .*

**7.2. The algebra  $\mathbf{K}_n^\epsilon$ .** Now we are back to the construction of an algebra  $\mathbf{K}_n^\epsilon$  out of  $\dot{\mathbf{K}}_n^\epsilon$  so that  $\dot{\mathbf{K}}_n^\epsilon$  is a modified (or idempotent) version of  $\mathbf{K}_n^\epsilon$ . Recall that

$$\mathbb{Z}_n^\epsilon = \{\lambda = (\lambda_i)_i \in \mathbb{Z}_n \mid \lambda_i = \lambda_{n+i} = \lambda_{-i}; \lambda_0, \lambda_{r+1} \in 2\mathbb{Z} + 1\}. \quad (7.2.1)$$

Let  $\hat{\mathbf{K}}_n^\epsilon$  be the  $\mathbb{Q}(v)$ -vector space of all (possibly infinite) linear combinations  $\sum_{A \in \tilde{\Xi}_n} \xi_A [A]$  for  $\xi_A \in \mathbb{Q}(v)$ ,  $[A] \in \dot{\mathbf{K}}_n^\epsilon$ , such that, for any  $\lambda \in \mathbb{Z}_n^\epsilon$ , the sets

$$\{A \in \tilde{\Xi}_n \mid \xi_A \neq 0, \text{row}_a(A) = \lambda\} \text{ and } \{A \in \tilde{\Xi}_n \mid \xi_A \neq 0, \text{col}_a(A) = \lambda\} \text{ are finite.} \quad (7.2.2)$$

We have an algebra structure on  $\hat{\mathbf{K}}_n^\epsilon$  which extends the one on  $\mathbf{K}_n^\epsilon$ :

$$\left( \sum_A \xi_A [A] \right) \cdot \left( \sum_B \eta_B [B] \right) = \sum_{A, B} \xi_A \eta_B [A][B].$$

Let  $\Xi_n^0$  be the subset of  $\Theta_n^0$  consisting of centro-symmetric matrices in  $\Theta_n^0$ . For any  $\nu \in \mathbb{Z}_n$ , we define  $\nu_\theta \in \mathbb{Z}_n^\epsilon$  by

$$(\nu_\theta)_i = \nu_i + \nu_{-i} + \sum_{k \in \mathbb{Z}} \delta_{i, k(r+1)}. \quad (7.2.3)$$

For each  $\mathbf{j} \in \mathbb{Z}_{r+2}$ ,  $A \in \Xi_n^0$ , we set

$$A(\mathbf{j}) = \sum_{\alpha \in \mathbb{Z}_n^\epsilon} v^{\mathbf{j} \bullet \alpha} [A + D_\alpha] \in \hat{\mathbf{K}}_n^\epsilon,$$

where

$$\mathbf{j} \bullet \alpha = j_0 \frac{\alpha_0 - 1}{2} + \sum_{i=1}^r j_i \alpha_i + j_{r+1} \frac{\alpha_{r+1} - 1}{2} \in \mathbb{Z}.$$

In particular, we have

$$A(\vec{0}) = \sum_{\alpha \in \mathbb{Z}_n^\epsilon} [A + D_\alpha], \quad 0(\mathbf{j}) = \sum_{\alpha \in \mathbb{Z}_n^\epsilon} v^{\mathbf{j} \bullet \alpha} [D_\alpha]. \quad (7.2.4)$$

Let  $\mathbf{K}_n^\epsilon$  be the  $\mathbb{Q}(v)$ -subspace of  $\hat{\mathbf{K}}_n^\epsilon$  spanned by

$$\mathfrak{B} = \{A(\mathbf{j}) \mid A \in \Xi_n^0, \mathbf{j} \in \mathbb{Z}_{r+2}\}.$$

**Proposition 7.2.** *The subspace  $\mathbf{K}_n^\epsilon$  is a subalgebra of  $\hat{\mathbf{K}}_n^\epsilon$  generated by  $A(\vec{0}), 0(\mathbf{j})$ , where  $A \in \Xi_n^0$  is tridiagonal and  $\mathbf{j} \in \mathbb{Z}_{r+2}$ .*

*Proof.* Denote by  $\mathbf{K}'_n$  the subalgebra of  $\hat{\mathbf{K}}_n^\epsilon$  generated by the elements specified in the proposition. We shall show that the space  $\mathbf{K}_n^\epsilon$  is closed under the left multiplication by the elements in the proposition. We give a detailed proof that  $B(\vec{0}) \cdot A(\mathbf{j}) \in \mathbf{K}_n^\epsilon$ , for tridiagonal  $B$  and arbitrary  $A$  in  $\Xi_n^0$ . (We shall skip a similar proof that  $0(\mathbf{j}_1)A(\mathbf{j})$ , for any  $A \in \Xi_n^0$  and  $\mathbf{j}_1, \mathbf{j} \in \mathbb{Z}_{r+2}$ .) It follows that  $\mathbf{K}'_n \subseteq \mathbf{K}_n^\epsilon$  by noting  $0(\mathbf{j}) \in \hat{\mathbf{K}}_n^\epsilon$ .

We have

$$B(\vec{0}) \cdot A(\mathbf{j}) = \sum_{\gamma \in \mathbb{Z}_n^\epsilon} v^{\mathbf{j} \bullet \gamma} [B + D_\beta][A + D_\gamma], \quad (7.2.5)$$



for a unique  $\beta$  satisfying  $\text{col}_c(B + D_\beta) = \text{row}_c(A + D_\gamma)$ . Let  $m_{B,A}^C \in \mathcal{A}$  be the structure constants in  $\dot{\mathbf{K}}_n^c$  such that  $[B] \cdot [A] = \sum_C m_{B,A}^C [C] \in \dot{\mathbf{K}}_n^c$ . By the multiplication formula in (5.3.1), we have

$$B(\vec{0}) \cdot A(\mathbf{j}) = \sum_{\gamma \in \mathbb{Z}_n^c} v^{\mathbf{j} \bullet \gamma} \sum_{T \in \Theta_{B+D_\beta, A+D_\gamma}} \sum_{S \in \Gamma_T} m_{B+D_\beta, A+D_\gamma}^{A^{(T-S)} + D_\gamma} [A^{(T-S)} + D_\gamma].$$

Note that  $\Theta_{B+D_\beta, A+D_\gamma} = \Theta_{B, A+D_\gamma}$  by construction. For any  $T \in \Theta_n$  such that  $T_\theta \leq_e A$ , we set  $\tau = \tau(B, A, T) \in \mathbb{Z}_n$  be such that

$$\tau_i = b_{i-1, i} - \text{row}_a(T)_i.$$

Therefore,  $T + D_\tau \in \Theta_{B, A+D_\gamma}$  for all  $\gamma \in \mathbb{Z}_n^c$  such that  $\tau_\theta \leq \gamma$ , and hence

$$B(\vec{0}) \cdot A(\mathbf{j}) = \sum_{\substack{T \in \Theta_n, T_\theta \leq_e A \\ \text{row}_a(T) \leq (b_{i-1, i})_i}} \sum_{S \in \Gamma_{T+D_\tau}} \sum_{\gamma \in \mathbb{Z}_n^c} v^{\mathbf{j} \bullet \gamma} m_{B+D_\beta, A+D_\gamma}^{A^{(T+D_\tau-S)} + D_\gamma} [A^{(T+D_\tau-S)} + D_\gamma].$$

We claim that

$$m_{B+D_\beta, A+D_\gamma}^{A^{(T+D_\tau-S)} + D_\gamma} = \sum_i v^{\mathbf{k}^{(i)} \bullet \gamma} m_{T, S}^{(i)} \quad (7.2.6)$$

for some  $\mathbf{k}^{(i)} \in \mathbb{Z}_{r+2}$ ,  $m_{T, S}^{(i)} \in \mathbb{Q}(v)$ . As a consequence, we obtain

$$\begin{aligned} B(\vec{0}) \cdot A(\mathbf{j}) &= \sum_{T, S, i} m_{T, S}^{(i)} \sum_{\gamma \in \mathbb{Z}_n^c} v^{(\mathbf{j} + \mathbf{k}^{(i)}) \bullet \gamma} [A^{(T+D_\tau-S)} + D_\gamma] \\ &= \sum_{T, S, i} m_{T, S}^{(i)} A^{(T+D_\tau, S)}(\mathbf{j} + \mathbf{k}^{(i)}) \in \mathbf{K}_n^c. \end{aligned}$$

Let us prove the claim (7.2.6). Note that if  $[A + D_\gamma] \in \mathbf{S}_{n, d}^c$ , the structure constants are given by

$$m_{B+D_\beta, A+D_\gamma}^{A^{(T+D_\tau-S)} + D_\gamma} = (v^2 - 1)^{n(S)} v^{\beta(A+D_\gamma, S, T+D_\tau) + \gamma(A+D_\gamma, S, T+D_\tau)} \llbracket A + D_\gamma; S; T + D_\tau \rrbracket.$$

A detailed calculation shows that the  $v$ -exponent  $\beta(A + D_\gamma, S, T + D_\tau) + \gamma(A + D_\gamma, S, T + D_\tau)$  is a polynomial in variables  $\gamma_k$  ( $0 \leq k \leq r + 1$ ) with total degree one. Also, by (4.3.4) we have

$$\begin{aligned} \llbracket A + D_\gamma; S; T + D_\tau \rrbracket &= M \cdot \prod_{1 \leq k \leq r} \prod_{i=1}^{(S+\widehat{T-S})_{\theta, kk}} (v^{2(\gamma_k - \tau_k - \tau_{-k} + i)} - 1) \\ &\quad \cdot \prod_{k \in \{0, r+1\}} \prod_{i=1}^{(S+\widehat{T-S})_{kk}} (v^{2(\gamma_k - 2\tau_k - 1 + 2i)} - 1), \end{aligned}$$

for some  $M \in \mathbb{Q}(v)$  that is independent of  $\gamma_k$ 's. Thus, (7.2.6) follows on the Schur algebra level, and hence it follows on the stabilization algebra level.

It remains to show that  $A(\mathbf{j})$ , for arbitrary  $A \in \Xi_n^0$ ,  $\mathbf{j} \in \mathbb{Z}_{r+2}$ , can be generated by these (expected) generators in the proposition. This follows from Theorem 5.16 and an induction on  $\Psi(A)$  (6.1.5). This then implies that  $\mathbf{K}'_n \supseteq \mathbf{K}_n^c$ . The proposition is proved.  $\square$

**7.3. The algebra  $\mathbf{K}_n^c$  as a subquotient.** In this subsection and the subsequent §7.4, we shall use [FLLLW, Section 9] substantially. In this preliminary subsection, we identify the algebra  $\mathbf{K}_n^c$  as a subquotient of a higher rank algebra generated by Chevalley generators.

Let

$$\check{n} = n + 2,$$

and

$$\tilde{\Xi}_{\check{n},1,0} = \{A \in \tilde{\Xi}_{\check{n}} \mid \text{row}_{\mathbf{a}}(A)_1 = \text{col}_{\mathbf{a}}(A)_1 = 0\}. \quad (7.3.1)$$

Recall from [FLLLW, §9.8] that there is a subalgebra  $\dot{\mathbf{K}}_{\check{n},1,0}^c$  of  $\dot{\mathbf{K}}_{\check{n}}^c$ :

$$\dot{\mathbf{K}}_{\check{n},1,0}^c = \text{Span}_{\mathbb{Q}(v)}\{[A] \in \dot{\mathbf{K}}_{\check{n}}^c \mid A \in \tilde{\Xi}_{\check{n},1,0}\}. \quad (7.3.2)$$

The algebra  $\dot{\mathbf{K}}_{\check{n},1,0}^c$  contains an ideal

$$\mathcal{I} = \text{Span}_{\mathbb{Q}(v)}\{[A] \in \dot{\mathbf{K}}_{\check{n},1,0}^c \mid a_{11} < 0\}.$$

For  $A \in \tilde{\Xi}_{\check{n}}$ , let  $\ddot{A}$  be the matrix in  $\tilde{\Xi}_{\check{n},1,0}$  obtained from  $A$  by inserting rows/columns of zeros between the 0th and  $\pm 1$ st rows/columns (mod  $n$ ). The assignment  $\tilde{\Xi}_{\check{n}} \rightarrow \tilde{\Xi}_{\check{n},1,0}$ ,  $A \mapsto \ddot{A}$ , induces an isomorphism

$$\tilde{\rho} : \dot{\mathbf{K}}_{\check{n}}^c \longrightarrow \dot{\mathbf{K}}_{\check{n},1,0}^c / \mathcal{I}, \quad [A] \mapsto [\ddot{A}] + \mathcal{I}. \quad (7.3.3)$$

Let  $\hat{\mathbf{K}}_{\check{n},1,0}^c$  be the subalgebra of  $\hat{\mathbf{K}}_{\check{n}}^c$  consisting of (possibly infinite) formal sums of the form  $\sum_{A \in \tilde{\Xi}_{\check{n},1,0}} \xi_A [A]$ , for  $\xi_A \in \mathbb{Q}(v)$ . In particular,  $\hat{\mathbf{K}}_{\check{n},1,0}^c$  contains the *restricted sums*

$$M[\mathbf{j}] = \sum_{\alpha \in \mathbb{Z}_{\check{n}}^{cr}} v^{\mathbf{j} \cdot \alpha} [M + D_{\alpha}], \quad \text{for } M \in \tilde{\Xi}_{\check{n},1,0}, \quad (7.3.4)$$

where we denote, for  $\mathbf{j} \in \mathbb{Z}_{r+2}$ ,

$$\tilde{\mathbf{j}} = (j_0, 0, j_1, j_2, \dots, j_{r+1}) \in \mathbb{Z}_{r+3}, \quad \mathbb{Z}_{\check{n}}^{cr} = \{\alpha = (\alpha_i) \in \mathbb{Z}_{\check{n}}^c \mid \alpha_1 = 0\}.$$

(We note that all infinite sums in this subsection are restricted.) The isomorphism  $\tilde{\rho}$  induces an isomorphism

$$\hat{\rho} : \hat{\mathbf{K}}_{\check{n}}^c \longrightarrow \hat{\mathbf{K}}_{\check{n},1,0}^c / \hat{\mathcal{I}}, \quad [A] \mapsto [\ddot{A}] + \hat{\mathcal{I}} \quad (7.3.5)$$

where  $\hat{\mathcal{I}}$  is the completion of  $\mathcal{I}$  in  $\hat{\mathbf{K}}_{\check{n}}^c$ . In particular,  $\hat{\rho}$  sends  $A(\mathbf{j})$  to  $\ddot{A}[\mathbf{j}] + \hat{\mathcal{I}}$ , for  $A \in \Xi_n^0$  and  $\mathbf{j} \in \mathbb{Z}_{r+2}$ .

Denote

$$\tilde{\Xi}'_{\check{n}} = \{A = (a_{ij}) \in \tilde{\Xi}_{\check{n}} \mid a_{ii} = 0, \text{ if } i \neq \pm 1 \pmod{\check{n}}\}.$$

By abuse of notation, we denote by  $\mathbf{Y}_{\check{n}}^c$  the subspace of  $\hat{\mathbf{K}}_{\check{n}}^c$  spanned by all restricted sums  $A[\mathbf{j}]$ , for  $A \in \tilde{\Xi}'_{\check{n}}$  and  $\mathbf{j} \in \mathbb{Z}_{r+2}$ .

**Lemma 7.3.** *The subspace  $\mathbf{Y}_{\check{n}}^c$  is a subalgebra of  $\hat{\mathbf{K}}_{\check{n}}^c$  generated by  $A[\vec{0}]$  and  $0[\mathbf{j}]$ , for all tridiagonal matrices  $A$  in  $\tilde{\Xi}'_{\check{n}}$  and  $\mathbf{j} \in \mathbb{Z}_{r+2}$ .*

*Proof.* The proof is essentially the same as that for Proposition 7.2 and we shall be brief. For the reader's convenience, we show that  $\mathbf{Y}_{\check{n}}^c$  is closed under multiplication on the left by  $S_{\alpha}[\vec{0}]$  for some tridiagonal matrices  $S_{\alpha}$ . By definition, we have

$$S_{\alpha}[\vec{0}] \cdot A[\mathbf{j}] = \sum_{\beta \in \mathbb{Z}_{\check{n}}^{cr}} [S_{\alpha} + D_{\beta'}] \cdot v^{\mathbf{j} \cdot \beta} [A + D_{\beta}],$$

where  $\beta'$  is uniquely determined by the condition  $\text{col}_a(S_\alpha + D_{\beta'}) = \text{row}_a(A + D_\beta)$ . Note that if  $\text{col}_a(S_\alpha)_1 \neq \text{row}_a(A)_1$  then  $S_\alpha[\vec{0}] \cdot A[\mathbf{j}] = 0$ . Otherwise, by a similar argument as that for Proposition 7.2, the product  $[S_\alpha + D_{\beta'}] \cdot v^{\mathbf{j} \cdot \beta} [A + D_\beta]$  is a linear combination of  $[A^{(T-S)} + D_\beta]$  whose coefficient is a linear combination of  $v^{\beta'' \cdot \mathbf{k}}$  for some  $\mathbf{k}$  and  $\beta'' = (\beta''_i) \in \mathbb{Z}_n^{cr}$  such that  $\beta''_i$  is the  $(i, i)$ th entry of  $A^{(T-S)} + D_\beta$  if  $i \neq \pm 1 \pmod n$  and 0 otherwise. By changing indices and taking sums over  $\beta'' \in \mathbb{Z}_n^{cr}$ , the product  $S_\alpha[\vec{0}] \cdot A[\mathbf{j}]$  is a linear combination of  $A'[\mathbf{k}]$  for some  $A' \in \tilde{\Xi}'_n$  and  $\mathbf{k} \in \mathbb{Z}_{r+2}$ .  $\square$

Let  $\mathbf{K}_{n,1,0}^\epsilon$  be the subspace of  $\mathbf{Y}_n^\epsilon$  spanned by  $\ddot{A}[\mathbf{j}]$  for all  $A \in \Xi_n^0, \mathbf{j} \in \mathbb{Z}_{r+2}$ . It follows by (7.3.5) that

$$\mathbf{K}_n^\epsilon \cong \hat{\rho}(\mathbf{K}_n^\epsilon) = (\mathbf{K}_{n,1,0}^\epsilon + \hat{\mathcal{I}})/\hat{\mathcal{I}}. \quad (7.3.6)$$

We now introduce a subalgebra of  $\mathbf{Y}_n^\epsilon$

$$\mathbf{K}_n^{\epsilon, ap} = \text{subalgebra generated by } E_{\theta, \check{n}}^{i, i+1}[\vec{0}], 0[\mathbf{j}], \forall 1 \leq i \leq \check{n}, \mathbf{j} \in \mathbb{Z}_{r+2}. \quad (7.3.7)$$

(Here the notation  $E_{\theta, \check{n}}^{i, i+1}$  is adapted from (2.3.6)–(2.3.7) with additional subscript  $\check{n}$  to indicate it lies in  $\tilde{\Xi}_n$ . Similar self-explanatory notation will be used below.)

**Lemma 7.4.** *For any tridiagonal  $A \in \Xi_n^0$ , the element  $\ddot{A}[\vec{0}]$  lies in  $\mathbf{K}_n^{\epsilon, ap} + \hat{\mathcal{I}}$ . In particular, we have  $\mathbf{K}_n^\epsilon \cong (\mathbf{K}_{n,1,0}^\epsilon + \hat{\mathcal{I}})/\hat{\mathcal{I}} \subseteq (\mathbf{K}_n^{\epsilon, ap} + \hat{\mathcal{I}})/\hat{\mathcal{I}}$ .*

*Proof.* The second part follows by definition and (7.3.6). So it remains to check the first statement in the lemma.

For any  $A \in \tilde{\Xi}_n$  such that  $A - \sum_{1 \leq i \leq n} \beta_i E_{\theta, n}^{i, i+1}$  is diagonal, we set

$$\ddot{\mathbf{f}}_A[\vec{0}] = \beta_0 E_{\theta, \check{n}}^{01}[\vec{0}] \cdot \beta_{n-1} E_{\theta, \check{n}}^{n, n+1}[\vec{0}] \cdot \beta_{n-1} E_{\theta, \check{n}}^{n+1, n+2}[\vec{0}] \cdot (\beta_{n-2} E_{\theta, \check{n}}^{n-1, n}[\vec{0}] \cdots \beta_1 E_{\theta, \check{n}}^{23}[\vec{0}] \cdot \beta_0 E_{\theta, \check{n}}^{12}[\vec{0}]).$$

By a similar argument as for [FLLLW, Lemma 9.1.2], we have

$$[\ddot{A} + D_\alpha] = \ddot{\mathbf{f}}_A[\vec{0}] \cdot [D_{\text{col}_a(\ddot{A} + D_\alpha)}] + \text{lower terms}, \quad \forall \alpha \in \mathbb{Z}_n^{cr}.$$

Therefore we have

$$\begin{aligned} \ddot{A}[\vec{0}] &= \sum_{\alpha \in \mathbb{Z}_n^{cr}} [A + D_\alpha] = \ddot{\mathbf{f}}_A[\vec{0}] \sum_{\alpha \in \mathbb{Z}_n^{cr}} [D_{\text{col}_a(\ddot{A} + D_\alpha)}] + \text{lower terms} \\ &= \ddot{\mathbf{f}}_A[\vec{0}] + \text{lower terms}. \end{aligned} \quad (7.3.8)$$

We now consider “lower terms” in (7.3.8). Since both  $\ddot{A}[\vec{0}]$  and  $\ddot{\mathbf{f}}_A[\vec{0}]$  are in  $\mathbf{Y}_n^\epsilon$ , these “lower terms” are also in  $\mathbf{Y}_n^\epsilon$ . Hence they are linear combinations of  $B[\mathbf{j}]$  for some  $B <_{\text{alg}} A$ . Since  $B[\mathbf{j}] = v^{-\check{\mathbf{j}} \cdot \text{col}_a(B)} B[\vec{0}] \cdot 0[\mathbf{j}]$ , by induction, it suffices to show any such  $B$  appearing in “lower terms” in (7.3.8) is equal to  $\ddot{C}$  up to  $\mathcal{I}$  for some  $C \in \Xi_n^0$ . It follows by (7.3.8) that  $B \in \tilde{\Xi}_{n,1,0}^\epsilon$ . Therefore we have  $B \in \mathcal{I}$  or  $B = \ddot{C}$  for some  $C \in \Xi_n^0$ . The lemma is proved.  $\square$

Now (in this paragraph) we repeat some of the above constructions in the affine type A setting quickly. We define

$$\tilde{\Theta}_{n,1,0} = \{A \in \tilde{\Theta}_n \mid \text{row}_a(A)_i = 0 = \text{col}_a(A)_i, i = \pm 1\}. \quad (7.3.9)$$

The algebra  $\dot{\mathbf{K}}_{\check{n}}$  contains a subalgebra  $\dot{\mathbf{K}}_{\check{n},1,0} = \text{Span}_{\mathbb{Q}(v)} \{[A] \in \dot{\mathbf{K}}_{\check{n}} \mid A \in \tilde{\Theta}_{\check{n},1,0}\}$ . For  $\mathbf{j} \in \mathbb{Z}_n$ , let  $\tilde{\mathbf{j}} \in \mathbb{Z}_{\check{n}}$  be uniquely determined by  $\tilde{\mathbf{j}}_1 = 0 = \tilde{\mathbf{j}}_{-1}, \tilde{\mathbf{j}}_0 = \mathbf{j}_0, \tilde{\mathbf{j}}_i = \mathbf{j}_{i-1}$  ( $2 \leq i \leq n-1$ ). For  $M \in \tilde{\Theta}_{\check{n}}$ , we define the restricted sum

$$M[\mathbf{j}]^a = \sum_{\substack{\alpha \in \mathbb{Z}_{\check{n}} \\ M + D_\alpha \in \tilde{\Theta}_{\check{n},1,0}}} v^{\tilde{\mathbf{j}} \cdot \alpha} [M + D_\alpha] \in \hat{\mathbf{K}}_{\check{n},1,0}.$$

As counterparts of  $\hat{\mathbf{K}}_{\check{n},1,0}^c$  (respectively,  $\mathbf{K}_{\check{n}}^{c,ap}$  and  $\mathbf{K}_{\check{n},1,0}^c$ ), we have their type A counterparts  $\hat{\mathbf{K}}_{\check{n},1,0}$  (respectively,  $\mathbf{K}_{\check{n}}^{ap}$  and  $\mathbf{K}_{\check{n},1,0}$ ). More precisely,  $\hat{\mathbf{K}}_{\check{n},1,0}$  is the subalgebra of  $\hat{\mathbf{K}}_{\check{n}}$  consisting of formal sums of the form  $\sum_{A \in \tilde{\Theta}_{\check{n},1,0}} \xi_A [A]$  ( $\xi_A \in \mathbb{Q}(v)$ ),  $\mathbf{K}_{\check{n}}^{ap}$  is the subalgebra of  $\hat{\mathbf{K}}_{\check{n},1,0}$  generated by  $E_{\check{n}}^{i,i+1}[\vec{0}]^a, E_{\check{n}}^{i+1,i}[\vec{0}]^a, 0[\mathbf{j}]^a$  ( $1 \leq i \leq \check{n}, \mathbf{j} \in \mathbb{Z}_n$ ), and  $\mathbf{K}_{\check{n},1,0}$  is the subalgebra of  $\hat{\mathbf{K}}_{\check{n},1,0}$  consisting of  $A[\mathbf{j}]^a$  for all  $A \in \tilde{\Theta}_{\check{n},1,0}$ . One can show that  $\mathbf{K}_{\check{n},1,0}$  is generated by  $E_{\check{n}}^{i,i+1}[\vec{0}]^a, E_{\check{n}}^{i+1,i}[\vec{0}]^a$  and  $0[\mathbf{j}]^a$  for  $2 \leq i \leq \check{n}-1, \mathbf{j} \in \mathbb{Z}_n$ .

**7.4. Comultiplication on  $\mathbf{K}_n^c$ .** In this subsection we shall show (in Theorem 7.5 and Theorem 7.8) that  $\mathbf{K}_n^c$  is a coideal subalgebra of  $\mathbf{K}_n$  and  $(\mathbf{K}_n, \mathbf{K}_n^c)$  forms a quantum symmetric pair. The construction in §7.3 allows us to study the comultiplication on  $\mathbf{K}_n^c$  via Chevalley generators.

Recall from [FLLLW, §9.6] a comultiplication  $\dot{\Delta}^c : \dot{\mathbf{K}}_n^c \longrightarrow \dot{\mathbf{K}}_n^c \otimes \dot{\mathbf{K}}_n$ . In the same way there is a comultiplication of  $\dot{\mathbf{K}}_{\check{n}}^c$ , whose restriction to  $\dot{\mathbf{K}}_{\check{n},1,0}^c$  is denoted by

$$\dot{\Delta}_{\check{n}}^c : \dot{\mathbf{K}}_{\check{n},1,0}^c \longrightarrow \dot{\mathbf{K}}_{\check{n},1,0}^c \otimes \dot{\mathbf{K}}_{\check{n},1,0}^c. \quad (7.4.1)$$

By [FLLLW, Lemma 9.3.1, Proposition 9.3.4], the two comultiplications  $\dot{\Delta}^c$  and  $\dot{\Delta}_{\check{n}}^c$  are compatible, i.e., the following diagram commutes:

$$\begin{array}{ccc} \dot{\mathbf{K}}_{\check{n},1,0}^c & \xrightarrow{\dot{\Delta}_{\check{n}}^c} & \dot{\mathbf{K}}_{\check{n},1,0}^c \otimes \dot{\mathbf{K}}_{\check{n},1,0}^c \\ q_1 \downarrow & & \downarrow q_2 \\ \dot{\mathbf{K}}_n^c & \xrightarrow{\dot{\Delta}^c} & \dot{\mathbf{K}}_n^c \otimes \dot{\mathbf{K}}_n, \end{array}$$

where both vertical maps are the canonical quotient maps. Let  $\mathcal{J} = \text{Ker}(q_2)$ . By diagram chasing, we have  $\dot{\Delta}_{\check{n}}^c(\mathcal{I}) \subseteq \mathcal{J}$ .

By passing to completions,  $\dot{\Delta}_{\check{n}}^c$  and  $\dot{\Delta}^c$  induce the following comultiplications:

$$\hat{\Delta}_{\check{n}}^c : \hat{\mathbf{K}}_{\check{n},1,0}^c \rightarrow \hat{\mathbf{K}}_{\check{n},1,0}^c \otimes \hat{\mathbf{K}}_{\check{n},1,0}^c \quad \text{and} \quad \hat{\Delta}^c : \hat{\mathbf{K}}_n^c \rightarrow \hat{\mathbf{K}}_n^c \otimes \hat{\mathbf{K}}_n. \quad (7.4.2)$$

**Theorem 7.5.** *The restriction of  $\hat{\Delta}^c$  in (7.4.2) to  $\mathbf{K}_n^c$  provides an algebra homomorphism*

$$\Delta^c = \hat{\Delta}^c|_{\mathbf{K}_n^c} : \mathbf{K}_n^c \longrightarrow \mathbf{K}_n^c \otimes \mathbf{K}_n.$$

*Proof.* Denote by  $\Delta_{\check{n}}^c$  the restriction of  $\hat{\Delta}_{\check{n}}^c$  to  $\mathbf{K}_{\check{n},1,0}^c$ . We first show  $\Delta_{\check{n}}^c(\mathbf{K}_{\check{n}}^{c,ap}) \subseteq \mathbf{K}_{\check{n}}^{c,ap} \otimes \mathbf{K}_{\check{n}}^{ap}$ . To this end, it suffices to check that  $\Delta_{\check{n}}^c(E_{\check{n}}^{i,i+1}[\vec{0}]) \in \mathbf{K}_{\check{n}}^{c,ap} \otimes \mathbf{K}_{\check{n}}^{ap}$  for  $1 \leq i \leq \check{n}$ .

Fix  $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_n^c$ , and let  ${}_a\dot{\mathbf{K}}_b^c$  be the subspace of  $\dot{\mathbf{K}}_n^c$  spanned by the standard basis elements  $[A]$  such that  $\text{row}_a(A) = \mathbf{a}$  and  $\text{col}_a(A) = \mathbf{b}$ . Recall from [FLLLW, §9.6] that

$$\Delta_{\check{n}}^c = (\Delta_{\check{n},\mathbf{a}',\mathbf{b}',\mathbf{a}'',\mathbf{b}''})_{\mathbf{a}',\mathbf{b}' \in \mathbb{Z}_{\check{n}}^c, \mathbf{a}'',\mathbf{b}'' \in \mathbb{Z}_n},$$

where  $\Delta_{\check{n}, \mathbf{a}', \mathbf{b}', \mathbf{a}'', \mathbf{b}''}^\zeta : \mathbf{a} \dot{\mathbf{K}}_{\mathbf{b}}^\zeta \rightarrow \mathbf{a}' \dot{\mathbf{K}}_{\mathbf{b}'}^\zeta \otimes_{\mathbf{a}''} \dot{\mathbf{K}}_{\mathbf{b}''}^\zeta$  is defined for any  $\mathbf{a}', \mathbf{b}' \in \mathbb{Z}_n^\zeta, \mathbf{a}'', \mathbf{b}'' \in \mathbb{Z}_n$  such that  $(\mathbf{a}', \mathbf{a}'') \models \mathbf{a}, (\mathbf{b}', \mathbf{b}'') \models \mathbf{b}$ , or equivalently, for

$$\mathbf{a} = \mathbf{a}' + \mathbf{a}_\theta'', \quad \mathbf{b} = \mathbf{b}' + \mathbf{b}_\theta''.$$

We have

$$\Delta_{\check{n}, \mathbf{a}', \mathbf{b}', \mathbf{a}'', \mathbf{b}''}^\zeta(E_\theta^{i, i+1}[\vec{0}]) = \sum_{\substack{\alpha \in \mathbb{Z}_n^\zeta \\ E_\theta^{i, i+1} + D_\alpha \in \tilde{\Xi}_{\check{n}, 1, 0}}} \Delta_{\check{n}, \mathbf{a}', \mathbf{b}', \mathbf{a}'', \mathbf{b}''}^\zeta([E_\theta^{i, i+1} + D_\alpha]).$$

We compute the contribution of  $\Delta_{\check{n}, \mathbf{a}', \mathbf{b}', \mathbf{a}'', \mathbf{b}''}^\zeta([E_\theta^{i, i+1} + D_\alpha])$  to each  $\mathbf{a}' \dot{\mathbf{K}}_{\mathbf{b}'}^\zeta \otimes_{\mathbf{a}''} \dot{\mathbf{K}}_{\mathbf{b}''}^\zeta$ . We have

$$\begin{aligned} \mathbf{a} &= \text{row}_{\mathbf{a}}(E_\theta^{i, i+1} + D_\alpha) = \epsilon_\theta^i + \alpha = \mathbf{a}' + \mathbf{a}_\theta'', \\ \mathbf{b} &= \text{col}_{\mathbf{a}}(E_\theta^{i, i+1} + D_\alpha) = \epsilon_\theta^{i+1} + \alpha = \mathbf{b}' + \mathbf{b}_\theta'', \end{aligned}$$

where

$$\epsilon^i = \begin{cases} 1 & \text{if } i \equiv j \pmod{\check{n}}, \\ 0 & \text{otherwise,} \end{cases} \in \mathbb{Z}_{\check{n}}$$

and  $\epsilon_\theta^i = \epsilon^i + \epsilon^{-i} \in \mathbb{Z}_n^\zeta$ . In other words, for any quadruple  $(\mathbf{a}', \mathbf{b}', \mathbf{a}'', \mathbf{b}'') \in \mathbb{Z}_n^\zeta \times \mathbb{Z}_n^\zeta \times \mathbb{Z}_n \times \mathbb{Z}_n$ ,  $\Delta_{\check{n}, \mathbf{a}', \mathbf{b}', \mathbf{a}'', \mathbf{b}''}^\zeta([E_\theta^{i, i+1} + D_\alpha])$  contributes to  $\mathbf{a}' \dot{\mathbf{K}}_{\mathbf{b}'}^\zeta \otimes_{\mathbf{a}''} \dot{\mathbf{K}}_{\mathbf{b}''}^\zeta$  if and only if  $(\mathbf{a}', \mathbf{b}', \mathbf{a}'', \mathbf{b}'')$  satisfies

$$\alpha = \mathbf{a}' + \mathbf{a}_\theta'' - \epsilon_\theta^i = \mathbf{b}' + \mathbf{b}_\theta'' - \epsilon_\theta^{i+1};$$

In this case the contribution is computed explicitly by [FLLW, Lemma 5.3.4] as

$$\Delta_{\check{n}, \mathbf{a}', \mathbf{b}', \mathbf{a}'', \mathbf{b}''}^\zeta([E_\theta^{i, i+1} + D_\alpha]) = \sum_{j=1}^3 g_j[B_j + D_{\beta^{(j)}}] \otimes {}^a[C_j + D_{\gamma^{(j)}}], \quad (7.4.3)$$

where  $\beta^{(j)} \in \mathbb{Z}_n^\zeta, \gamma^{(j)} \in \mathbb{Z}_{\check{n}}, B_j \in \Xi_{\check{n}}^0, C_j \in \Theta_{\check{n}}^0$  and  $g_j \in \mathbb{Q}(v)$  are give by

$j$	$B_j$	$C_j$	$\mathbf{a}'$	$\mathbf{b}'$	$\mathbf{a}''$	$\mathbf{b}''$
1	$E_\theta^{i, i+1}$	0	$\epsilon_\theta^i + \beta^{(1)}$	$\epsilon_\theta^{i+1} + \beta^{(1)}$	$\gamma^{(1)}$	$\gamma^{(1)}$
2	0	$E^{i, i+1}$	$\beta^{(2)}$	$\beta^{(2)}$	$\gamma^{(2)} + \epsilon^i$	$\gamma^{(2)} + \epsilon^{i+1}$
3	0	$E^{-i, -i-1}$	$\beta^{(3)}$	$\beta^{(3)}$	$\gamma^{(3)} + \epsilon^{-i}$	$\gamma^{(3)} + \epsilon^{-i-1}$

$$g_1 = \begin{cases} v^{(\epsilon^{\check{n}-i} - \epsilon^{\check{n}-1-i}) \cdot \gamma^{(1)}} & \text{if } \alpha = \beta^{(1)} + \gamma_\theta^{(1)}, \\ 0 & \text{otherwise.} \end{cases} \quad (7.4.4)$$

$$g_2 = \begin{cases} v^{-\delta_{i,0}} v^{(\epsilon^i - \epsilon^{i+1}) \cdot \beta^{(2)}} v^{(\epsilon^{\check{n}-i} - \epsilon^{\check{n}-1-i}) \cdot \gamma^{(2)}} & \text{if } \alpha = \beta^{(2)} + \gamma_\theta^{(2)}, \\ 0 & \text{otherwise.} \end{cases} \quad (7.4.5)$$

$$g_3 = \begin{cases} 1 & \text{if } \alpha = \beta^{(3)} + \gamma_\theta^{(3)}, \\ 0 & \text{otherwise.} \end{cases} \quad (7.4.6)$$

Next we show that  $\Delta_{\check{n}}^\zeta(E_\theta^{i, i+1}(\vec{0})) \in \mathbf{K}_{\check{n}, 1, 0}^\zeta \otimes \mathbf{K}_{\check{n}, 1, 0}$  by assembling these together. Note that

$$B[\mathbf{j}] \otimes C[\mathbf{k}]^a = \sum_{\substack{\beta \in \mathbb{Z}_n^\zeta \\ B + D_\beta \in \tilde{\Xi}_{\check{n}, 1, 0}}} \sum_{\substack{\gamma \in \mathbb{Z}_{\check{n}} \\ C + D_\gamma \in \tilde{\Theta}_{\check{n}, 1, 0}}} v^{\tilde{\mathbf{j}} \cdot \beta} v^{\tilde{\mathbf{k}} \cdot \gamma} [B + D_\beta] \otimes {}^a[C + D_\gamma].$$

Fix  $B = B_j, C = C_j, \beta \in \mathbb{Z}_{\check{n}}, \gamma \in \mathbb{Z}_{\check{n}}$  such that  $B + D_\beta \in \tilde{\Xi}_{\check{n},1,0}$  and  $C + D_\gamma \in \tilde{\Theta}_{\check{n},1,0}$ . There is a unique  $\alpha \in \mathbb{Z}_{\check{n}}^c$  determined by either (7.4.4), (7.4.5) or (7.4.6) such that  $\Delta_{\check{n},\mathbf{a}',\mathbf{b}',\mathbf{a}'',\mathbf{b}''}^c([E_\theta^{i,i+1} + D_\alpha])$  contributes to  $B[\mathbf{j}] \otimes C[\mathbf{k}]^a$ . It can be verified that  $E_\theta^{i,i+1} + D_\alpha \in \tilde{\Xi}_{\check{n},1,0}$ , and the contribution for such  $\alpha$  is counted. Therefore we have

$$\begin{aligned} \Delta_{\check{n}}^c(E_\theta^{i,i+1}[\vec{0}]) &= E_\theta^{i,i+1}[\vec{0}] \otimes 0[\epsilon^{\check{n}-i} - \epsilon^{\check{n}-1-i}]^a \\ &\quad + 0[\epsilon^i - \epsilon^{i+1}] \otimes E^{i,i+1}[\epsilon^{\check{n}-i} - \epsilon^{\check{n}-1-i}]^a + 0[\vec{0}] \otimes E^{-i,-i-1}[\vec{0}]^a. \end{aligned}$$

This completes the proof that  $\Delta_{\check{n}}^c(\mathbf{K}_{\check{n}}^{c,ap}) \subseteq \mathbf{K}_{\check{n}}^{c,ap} \otimes \mathbf{K}_{\check{n}}^{ap}$ .

Therefore, for any  $\ddot{A}[\vec{0}] \in \mathbf{K}_{\check{n},1,0}^c$ , by Lemma 7.4 we have  $\Delta_{\check{n}}^c(\ddot{A}[\vec{0}]) \in \mathbf{K}_{\check{n}}^{c,ap} \otimes \mathbf{K}_{\check{n}}^{ap} + \mathcal{J}$ . We now consider the summand in  $\mathbf{K}_{\check{n}}^{c,ap} \otimes \mathbf{K}_{\check{n}}^{ap}$ . Since  $\text{row}_a(\ddot{A})_1 = \text{col}_a(\ddot{A})_1 = 0$ , by [FLLLW, Proposition 9.3.4], the image of  $\Delta_{\check{n},\mathbf{a}',\mathbf{b}',\mathbf{a}'',\mathbf{b}''}^c(\ddot{A}[\vec{0}])$  in  $\mathbf{K}_{\check{n}}^{c,ap} \otimes \mathbf{K}_{\check{n}}^{ap}$  is zero unless  $a'_1 = 0, b' = 1$  and  $a''_1 = 0 = b''_1$ . This implies that  $\Delta_{\check{n}}^c(\ddot{A}[\vec{0}]) \in \mathbf{K}_{\check{n},1,0}^c \otimes \mathbf{K}_{\check{n},1,0} + \mathcal{J}$ . Finally, note that  $\Delta_{\check{n}}^c(\mathcal{I}) \subseteq \mathcal{J}$ . The proposition follows.  $\square$

The coassociativity of  $\dot{\Delta}^c$  descends to a similar one on  $\Delta^c$  as formulated below.

**Corollary 7.6.** *The comultiplication  $\Delta^c$  on  $\mathbf{K}_n^c$  satisfies the coassociativity property, i.e.,  $(1 \otimes \Delta)\Delta^c = (\Delta^c \otimes 1)\Delta^c$ .*

**Proposition 7.7.** *There is a natural injective algebra homomorphism  $\iota^c : \mathbf{K}_n^c \longrightarrow \mathbf{K}_n$ .*

*Proof.* The homomorphism  $\iota^c : \mathbf{K}_n^c \longrightarrow \mathbf{K}_n$  which we shall construct should be regarded as a degenerate variant of the comultiplication  $\Delta^c : \mathbf{K}_n^c \rightarrow \mathbf{K}_n^c \otimes \mathbf{K}_n$ , and the proof here is similar to that for Theorem 7.5. For the reader's convenience, we shall sketch the proof.

By setting  $d' = 0$  in [FLLLW, Lemma 9.3.1], we have the following commutative diagram

$$\begin{array}{ccc} \mathbf{S}_{n,d}^c & \xrightarrow{\iota_n} & \mathbf{S}_{n,d} \\ \rho_d \downarrow & & \downarrow \rho_{d''} \\ \mathbf{U}_{\check{n},d}^c & \xrightarrow{\iota_{\check{n}}} & \mathbf{U}_{\check{n},d} \end{array}$$

By setting  $d' = 0$  in [FLLLW, Proposition 9.3.4], we have a similar result for  $\iota_n$  or  $\iota_{\check{n}}$  instead of  $\Delta^c$ . Moreover, we have the following commutative diagram

$$\begin{array}{ccc} \dot{\mathbf{K}}_n^c & \xrightarrow{\iota_n} & \dot{\mathbf{K}}_n \\ q_1 \uparrow & & \uparrow q_2 \\ \dot{\mathbf{K}}_{\check{n},1,0}^c & \xrightarrow{\iota_{\check{n}}} & \dot{\mathbf{K}}_{\check{n},1,0} \end{array}$$

Recall that  $\mathcal{I} = \text{Span}_{\mathbb{Q}(v)}\{[A] \in \dot{\mathbf{K}}_{\check{n},1,0}^c \mid a_{11} < 0\}$ . We also set  $\mathcal{L} = \text{Span}_{\mathbb{Q}(v)}\{[A] \in \dot{\mathbf{K}}_{\check{n},1,0} \mid a_{11} < 0\}$ . By diagram chasing, we have  $\iota_{\check{n}}(\mathcal{I}) \subseteq \mathcal{L}$ .

We now show that  $\iota_{\check{n}}(\mathbf{K}_{\check{n}}^{c,ap}) \subseteq \mathbf{K}_{\check{n}}^{ap}$ . By the definition of  $\mathbf{K}_{\check{n}}^{c,ap}$  in (7.3.7), it suffices to show that  $\iota_{\check{n}}(E_{\theta,\check{n}}^{i,i+1}[\vec{0}]) \in \mathbf{K}_{\check{n}}^{ap}$ . We observe that for any  $\lambda \in \mathbb{Z}_{\check{n}}^c$ ,  $\iota_{\check{n}}(D_\lambda) = D_{\lambda'}$  for some  $\lambda' \in \mathbb{Z}_{\check{n}}$ .

Hence, we have

$$\begin{aligned}
 \iota_{\check{n}}(E_{\theta}^{i,i+1}[\vec{0}]) &= \sum_{\alpha \in \mathbb{Z}_{\check{n}}^{\epsilon}, E_{\theta}^{i,i+1} + D_{\alpha} \in \tilde{\Xi}_{\check{n},1,0}} \iota_{\check{n}}([E_{\theta}^{i,i+1} + D_{\alpha}]) \\
 &= \sum_{\alpha \in \mathbb{Z}_{\check{n}}^{\epsilon}, E_{\theta}^{i,i+1} + D_{\alpha} \in \tilde{\Xi}_{\check{n},1,0}} \iota_{\check{n}}(f_i D_{\alpha'}) \\
 &= \sum_{\alpha \in \mathbb{Z}_{\check{n}}^{\epsilon}, E_{\theta}^{i,i+1} + D_{\alpha} \in \tilde{\Xi}_{\check{n},1,0}} e_{n-1-i} D_{\beta} + v^{\delta_{i0}} f_i k_{n-1-i} D_{\beta} \\
 &= E^{n-i, n-i-1}[\vec{0}]^{\mathbf{a}} + v^{\delta_{i0}} E^{i,i+1}[\vec{0}]^{\mathbf{a}} 0[\epsilon^{\check{n}-i} - \epsilon^{\check{n}-1-i}]^{\mathbf{a}} \in \mathbf{K}_{\check{n}}^{ap},
 \end{aligned}$$

where  $\alpha' \in \mathbb{Z}_{\check{n}}^{\epsilon}$  and  $\beta \in \mathbb{Z}_{\check{n}}$  are determined by  $\text{col}_{\mathbf{a}}(E_{\theta}^{i,i+1} + D_{\alpha}) = \alpha'$  and  $\iota_{\check{n}}(D_{\alpha'}) = D_{\beta}$ , respectively.

By tracking  $\text{row}_{\mathbf{a}}(A)_1$  and  $\text{col}_{\mathbf{a}}(A)_1$ , we have  $\iota_{\check{n}}(\ddot{A}[\vec{0}]) \in \mathbf{K}_{\check{n},1,0}$ , i.e.  $\iota_{\check{n}}(\mathbf{K}_{\check{n},1,0}^{\epsilon}) \subseteq \mathbf{K}_{\check{n},1,0}$ . Therefore  $\iota_{\check{n}}(\mathbf{K}_{\check{n},1,0}^{\epsilon}/\mathcal{I}) \subseteq \mathbf{K}_{\check{n},1,0}/\mathcal{L}$ , and this induces the desired homomorphism  $\iota^{\epsilon} : \mathbf{K}_n^{\epsilon} \rightarrow \mathbf{K}_n$ . The injectivity of  $\iota^{\epsilon}$  follows from the injectivity of  $\iota_{\check{n}}$ . The proposition is proved.  $\square$

Recall the notation of quantum symmetric pairs as defined in [Le02] for finite type and in [Ko14] for Kac-Moody type. We rephrase Theorem 7.5 and Proposition 7.7 as follows. (We recall by Proposition 7.1 that the algebra  $\mathbf{K}_n$  is isomorphic to the quantum affine  $\mathfrak{gl}_n$ .)

**Theorem 7.8.** *The pair  $(\mathbf{K}_n, \mathbf{K}_n^{\epsilon})$  forms a quantum symmetric pair of affine type.*

## 8. STABILIZATION ALGEBRAS ARISING FROM OTHER SCHUR ALGEBRAS

In this section we formulate three more variants (denoted by  $\mathfrak{u}, \mathfrak{v}, \mathfrak{w}$ ) of affine Schur algebras and their corresponding stabilization algebras. We construct the standard, monomial, and stably canonical bases of these algebras. We will present more details for the type  $\mathfrak{v}$ . We will merely be formulating the main statements for the remaining types  $\mathfrak{u}$  and  $\mathfrak{w}$ .

Recall  $n = 2r + 2$ , and we now set

$$\mathfrak{n} = n - 1 = 2r + 1 \quad (r \geq 1).$$

**8.1. Affine Schur algebras of type  $\mathfrak{v}$ .** Recall  $\Xi_{n,d}$  from (2.3.5). Let

$$\Xi_{\mathfrak{n},d}^{\mathfrak{v}} = \{A \in \Xi_{n,d} \mid \text{row}_{\mathbf{c}}(A)_0 = 0 = \text{row}_{\mathbf{c}}(A)_0\}. \quad (8.1.1)$$

The additional condition for  $A \in \Xi_{n,d}$  to be in  $\Xi_{\mathfrak{n},d}^{\mathfrak{v}}$  can be equivalently reformulated as

$$\text{row}_{\mathbf{a}}(A)_i = \begin{cases} 1 & \text{if } i \equiv 0 \pmod{n}, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, a general element  $A \in \Xi_{n,d}^{\nu_j}$  (8.1.1) is of the form below (where  $r\#\backslash c\#$  stands for row  $\#$  and column  $\#$  of the matrix):

$$A = \begin{array}{c|cccccccc} r\#\backslash c\# & \cdots & -1 & 0 & 1 & \cdots & n-1 & n & n+1 & \cdots \\ \hline \vdots & \ddots & & \vdots & & & & \vdots & & \\ -1 & & a_{n-1,n-1} & 0 & a_{-1,1} & & * & 0 & * & \\ \hline 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \hline 1 & & a_{1,-1} & 0 & a_{11} & & * & 0 & * & \\ \vdots & & & \vdots & & \ddots & & \vdots & & \\ n-1 & & * & 0 & * & & a_{n-1,n-1} & 0 & a_{-1,1} & \\ \hline n & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots \\ \hline n+1 & & * & 0 & * & & a_{1,-1} & 0 & a_{11} & \\ \vdots & & & \vdots & & & & \vdots & & \ddots \end{array} \quad (8.1.2)$$

Recall  $\Lambda = \Lambda_{r,d}$  (2.2.1). Let

$$\begin{aligned} \Lambda^{\nu_j} &= \{\lambda = (\lambda_i)_{i \in \mathbb{Z}} \in \Lambda \mid \lambda_0 = 0\}, \\ \mathcal{D}_{n,d}^{\nu_j} &= \{(\lambda, g, \mu) \mid \lambda, \mu \in \Lambda^{\nu_j}, g \in \mathcal{D}_{\lambda\mu}\}. \end{aligned}$$

The lemma below is the  $\nu_j$ -analog of Lemma 2.7, which follows by a similar argument.

**Lemma 8.1.** *The map  $\kappa^{\nu_j} : \mathcal{D}_{n,d}^{\nu_j} \longrightarrow \Xi_{n,d}^{\nu_j}$  sending  $(\lambda, g, \mu)$  to  $(|R_i^\lambda \cap gR_j^\mu|)$  is a bijection.*

Now we define the *affine  $q$ -Schur algebra of type  $\nu_j$*  as

$$\mathbf{S}_{n,d}^{\nu_j} = \text{End}_{\mathbf{H}}\left(\bigoplus_{\lambda \in \Lambda^{\nu_j}} x_\lambda \mathbf{H}\right). \quad (8.1.3)$$

By definition the algebra  $\mathbf{S}_{n,d}^{\nu_j}$  is naturally a subalgebra of  $\mathbf{S}_{n,d}^c$ . Moreover, both  $\{e_A \mid A \in \Xi_{n,d}^{\nu_j}\}$  and  $\{[A] \mid A \in \Xi_{n,d}^{\nu_j}\}$  are bases of  $\mathbf{S}_{n,d}^{\nu_j}$  as a free  $\mathcal{A}$ -module. Note that although Algorithm 5.15 applies to arbitrary  $A \in \Xi_{n,d}^{\nu_j} (\subset \Xi_{n,d})$ , the resulting matrices  $A^{(i)}$  do not lie in  $\Xi_{n,d}^{\nu_j}$  in general. In order to define a monomial basis for  $\mathbf{S}_{n,d}^{\nu_j}$ , we need a modified matrix interpretation by collapsing those fixed rows and columns indexed by  $n\mathbb{Z}$  in (8.1.1). Let

$$\mathbb{Z}^{\nu_j} = \mathbb{Z} \backslash n\mathbb{Z},$$

and

$$\begin{aligned} \check{\Xi}_{n,d}^{\nu_j} &= \left\{ A = (a_{ij}) \in \text{Mat}_{\mathbb{Z}^{\nu_j} \times \mathbb{Z}^{\nu_j}}(\mathbb{N}) \mid a_{-i,-j} = a_{ij} = a_{i+n,j+n}, \forall i, j \in \mathbb{Z}^{\nu_j}; \right. \\ &\quad \left. a_{00}, a_{r+1,r+1} \text{ are odd}; \sum_{1 \leq i \leq n-1} \sum_{j \in \mathbb{Z}^{\nu_j}} a_{ij} = D-1 \right\}. \end{aligned} \quad (8.1.4)$$



That is, a general element  $A \in \check{\Xi}_{n,d}^{ij}$  (8.1.4) is of the form below:

$$A = \begin{array}{c|c|c|c|c|c} r\#\backslash c\# & \cdots & -1 & 1 & \cdots & n-1 & n+1 & \cdots \\ \hline \vdots & \ddots & & & & & & \\ -1 & & a_{11} & a_{1,-1} & & * & & * \\ 1 & & a_{1,-1} & a_{11} & & * & & * \\ \vdots & & & & \ddots & & & \\ n-1 & & * & * & & a_{11} & a_{1,-1} & \\ n+1 & & * & * & & a_{1,-1} & a_{11} & \\ \vdots & & & & & & & \ddots \end{array} \quad (8.1.5)$$

Here the solid lines replace the dashed stripes in (8.1.2). Let  $f_c^{ij} : \Xi_{n,d}^{ij} \rightarrow \check{\Xi}_{n,d}^{ij}$  be the “collapsing” map by removing the columns/rows indexed by  $n\mathbb{Z}$ , and let  $f_e^{ij} : \check{\Xi}_{n,d}^{ij} \rightarrow \Xi_{n,d}^{ij}$  be the expanding map by inserting the fixed columns/rows indexed by  $n\mathbb{Z}$  as in (8.1.2). Clearly  $f_c^{ij}$  and  $f_e^{ij}$  are inverse maps to each other, and hence are bijections.

**Example 8.2.** We shall follow the conventions of dashed/solids lines in (8.1.2) and (8.1.5). Let  $n = 6$  (and hence  $r = 2$ ), and

$$A = \left[ \begin{array}{cccccc|cc|cc} 3 & & & & & & & & & & \\ * & 4 & & & & & & & & & \\ 3 & * & 5 & & & & & & & & \\ & 2 & * & 0 & 1 & & & & & & \\ \hline & & 0 & 1 & 0 & & & & & & \\ \hline & & 1 & 0 & * & 2 & & & & & \\ & & & & 5 & * & 3 & & & & \\ & & & & & 4 & * & & & & \\ & & & & & & 3 & & & & \end{array} \right] \in \Xi_{n,d}^{ij}, \quad \text{centered at the } (0,0)\text{th entry.}$$

That is,  $A^* = E_\theta^{1,-1} + 2E_\theta^{12} + 3E_\theta^{23} + 4E_\theta^{32} + 5E_\theta^{21}$  (cf. (4.4.1) for notation). Applying Algorithm 5.15, we get

$$[A^{(1)}][A^{(2)}] = [A] + \text{lower terms} \in \mathbf{S}_{n,d}^c,$$

where

$$A^{(1)} = \left[ \begin{array}{cccccc|cc|cc} & & & & & & & & & & \\ & * & & & & & & & & & \\ & & * & & & & & & & & \\ & & & * & 1 & & & & & & \\ \hline & & & & 1 & & & & & & \\ \hline & & & & 1 & & & & * & & \\ & & & & & & & & * & & \\ & & & & & & & & & * & \end{array} \right] \in \Xi_{n,d}, \quad A^{(2)} = \left[ \begin{array}{cccccc|cc|cc} 3 & & & & & & & & & & \\ * & 4 & & & & & & & & & \\ 3 & * & 5 & & & & & & & & \\ & 2 & * & 0 & & & & & & & \\ \hline & & 1 & 1 & 1 & & & & & & \\ \hline & & & 0 & * & 2 & & & & & \\ & & & & 5 & * & 3 & & & & \\ & & & & & 4 & * & & & & \\ & & & & & & 3 & & & & \end{array} \right] \in \Xi_{n,d}.$$

Note that neither  $A^{(1)}$  nor  $A^{(2)}$  lies in  $\Xi_{n,d}^{ij}$ . On the other hand, we have

$$f_c^{ij}(A) = \left[ \begin{array}{ccc|ccc} 3 & & & & & \\ * & 4 & & & & \\ 3 & * & 5 & & & \\ & 2 & * & 1 & & \\ \hline & & & 1 & * & 2 \\ & & & & 5 & * & 3 \\ & & & & & 4 & * \\ & & & & & & 3 \end{array} \right] \in \check{\Xi}_{n,d}^{ij}.$$

That is,  $f_c^{ij}(A)^\star = E_\theta^{1,-1} + 2E_\theta^{12} + 3E_\theta^{23} + 4E_\theta^{32} + 5E_\theta^{21}$ .

**8.2. Monomial and canonical bases for  $\mathbf{S}_{n,d}^{ij}$ .** While  $A \in \Xi_{n,d}^{ij}$  is not tridiagonal,  $f_c^{ij}(A) \in \check{\Xi}_{n,d}^{ij}$  is tridiagonal in the following sense:  $X = (x_{ij}) \in \check{\Xi}_{n,d}^{ij}$  is called tridiagonal if

$$x_{ij} = 0 \quad \text{if} \quad |\tau(i) - \tau(j)| > 1, \quad (8.2.1)$$

where

$$\tau : \mathbb{Z}^{ij} \longrightarrow \mathbb{Z} \quad (8.2.2)$$

is the order-preserving bijection determined by setting  $\tau(1) = 1$  (and thus  $\tau(-1) = 0$ , and so on). We claim that the set  $\{[A] \in \mathbf{S}_{n,d}^{ij} \mid A \in \Xi_{n,d}^{ij} \text{ such that } f_c^{ij}(A) \in \check{\Xi}_{n,d}^{ij} \text{ is tridiagonal}\}$  is a generating set (even though in this case  $A \in \Xi_{n,d}^{ij}$  may not be tridiagonal). We shall now give an algorithm in the framework of  $\Xi_{n,d}^{ij}$  (compare Algorithm 5.15).

**Algorithm 8.3.** For any  $A \in \Xi_{n,d}^{ij}$ , we define matrices  ${}^i A^{(1)}, \dots, {}^i A^{(x)} \in \Xi_{n,d}^{ij}$  as follows:

- (1) Initialization: set  $t = 0$ , and set  $C^{(0)} = f_c^{ij}(A) \in \check{\Xi}_{n,d}^{ij}$ .
- (2) If  $C^{(t)} = (c_{ij}^{(t)}) \in \check{\Xi}_{n,d}^{ij}$  is a tridiagonal matrix (cf. (8.2.1)), then set  $x = t + 1$ ,  $A^{(x)} = C^{(t)}$ , and the algorithm terminates.

Otherwise, set  $k = \max \{|\tau(i) - \tau(j)| \mid c_{ij}^{(t)} \neq 0\} \geq 2$ , and

$$\mathbf{T}^{(t)} = \sum_{i=1}^{n-1} c_{i,\tau^{-1}(\tau(i)+k)}^{(t)} E_\theta^{i,\tau^{-1}(\tau(i)+k)}.$$

- (3) Define matrices

$$A^{(t+1)} = \sum_{i=1}^{n-1} c_{i,\tau^{-1}(\tau(i)+k)}^{(t)} E_\theta^{i,\tau^{-1}(\tau(i)+1)} + \text{a diagonal determined by (5.6.1),}$$

$$C^{(t+1)} = C^{(t)} - \mathbf{T}_\theta^{(t)} + \widetilde{\mathbf{T}}_\theta^{(t)},$$

where  $\widetilde{X}$  is the matrix obtained from  $X$  by shifting every entry down by one row; also see (2.3.7) for notation.

- (4) Set  ${}^i A^{(t+1)} = f_e^{ij}(A^{(t+1)})$ . Increase  $t$  by one and then go to Step (2).

**Theorem 8.4.** *For each  $A \in \Xi_{n,d}^{ij}$ , the matrices  ${}^i A^{(j)} \in \Xi_{n,d}^{ij}$ , for  $j = 1, \dots, x$ , in Algorithm 8.3 satisfy that*

$$[{}^i A^{(1)}][{}^i A^{(2)}] \cdots [{}^i A^{(x)}] = [A] + \text{lower terms} \in \mathbf{S}_{n,d}^{ij}. \quad (8.2.3)$$

*Proof.* Applying Algorithm 5.15 on  ${}^i A^{(t)}$ , we obtain tridiagonal matrices  $D_1^{(t)}$  and  $D_2^{(t)}$  in  $\Xi_{n,d}$  satisfying that

$$[D_1^{(t)}][D_2^{(t)}] = [{}^i A^{(t)}] + \text{lower terms} \in \mathbf{S}_{n,d}^c,$$

which implies from construction that

$$[{}^i A^{(1)}][{}^i A^{(2)}] \cdots [{}^i A^{(x)}] = [A] + \text{lower terms} \in \mathbf{S}_{n,d}^c.$$

It remains to show that each lower term  $[D]$  in the above identity lies in  $\mathbf{S}_{n,d}^{ij}$ . By definition of multiplication on  $\mathbf{S}_{n,d}^{ij}$  we have  $\text{row}_c(D) = \text{row}_c(A)$  and  $\text{col}_c(D) = \text{col}_c(A)$ . In particular,  $D \in \Xi_{n,d}^{ij}$  and hence  $[D] \in \mathbf{S}_{n,d}^{ij}$ .  $\square$

**Example 8.5.** We illustrate how Algorithm 8.3 and proof of Theorem 8.4 work. Let  $n = 4$  (and hence  $r = 1$ ), and

$$A = \left[ \begin{array}{cccc|cccc} 4 & 0 & 0 & 2 & 1 & & & \\ & 0 & 1 & 0 & & & & \\ 1 & 2 & 0 & 0 & 4 & & & \\ & & & 3 & 1 & 3 & & \\ & & & & 4 & 0 & & \end{array} \right] \in \Xi_{n,d}^{ij}.$$

We have

$$C^{(0)} = \left[ \begin{array}{cc|cc} 4 & 0 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ & & 3 & 1 & 3 \\ & & & 4 & 0 \end{array} \right] \in \check{\Xi}_{n,d}^{ij}.$$

In this case, we have  $k = 2$  and hence

$$T^{(0)} = \left[ \begin{array}{c|c} 0 & 1 \\ 1 & 0 \\ & 0 \\ & 0 \end{array} \right], \quad A^{(1)} = \left[ \begin{array}{c|c} 6 & 1 \\ 1 & 6 \\ & 7 \\ & 6 \end{array} \right], \quad C^{(1)} = \left[ \begin{array}{c|ccc} 0 & 2 & & & \\ 2 & 0 & 5 & & \\ & 3 & 1 & 3 & \\ & & 5 & 0 & \end{array} \right] \in \check{\Xi}_{n,d}^{ij}.$$

Now  $C^{(1)}$  is tridiagonal in  $\check{\Xi}_{n,d}^{ij}$  and hence  $A^{(2)} = C^{(1)}$ , which yields that

$${}^i A^{(1)} = \left[ \begin{array}{cccc|cccc} 6 & 0 & 1 & & & & & \\ 0 & 1 & 0 & & & & & \\ 1 & 0 & 6 & & & & & \\ & & & 7 & & & & \\ & & & & 6 & & & \end{array} \right], \quad {}^i A^{(2)} = \left[ \begin{array}{cccc|cccc} 0 & 0 & 2 & & & & & \\ 0 & 1 & 0 & & & & & \\ 2 & 0 & 0 & 5 & & & & \\ & & & 3 & 1 & 3 & & \\ & & & & 5 & 0 & & \end{array} \right] \in \Xi_{n,d}^{ij}.$$

Therefore, we have

$$D_1^{(1)} = \left[ \begin{array}{ccc|ccc} 6 & 1 & & & & \\ & 1 & & & & \\ & & 1 & 6 & & \\ & & & & 7 & \end{array} \right], \quad D_2^{(1)} = \left[ \begin{array}{ccc|ccc} 0 & 0 & & & & \\ 1 & 1 & 1 & & & \\ & & 6 & & & \\ & & & 7 & & \end{array} \right], \quad D_1^{(2)} = \left[ \begin{array}{ccc|ccc} 0 & 2 & & & & \\ 0 & 1 & 0 & & & \\ & 2 & 5 & & & \\ & & & 7 & & \end{array} \right], \quad D_2^{(2)} = \left[ \begin{array}{ccc|ccc} 0 & 0 & & & & \\ 2 & 1 & 2 & & & \\ & 0 & 0 & 5 & & \\ & & & 3 & 1 & \end{array} \right].$$

Modifying (8.2.3), for  $A \in \Xi_{n,d}^{ij}$ , we define

$$m_A = \{{}^i A^{(1)}\} \{{}^i A^{(2)}\} \cdots \{{}^i A^{(x)}\}. \quad (8.2.4)$$

Recall from Theorem 5.8 the canonical basis  $\mathfrak{B}_{n,d}^c$  for  $\mathbf{S}_{n,d}^c$ . The following is the the  $ij$ -counterpart of Theorems 5.8 and 5.18 (for  $\mathbf{S}_{n,d}^c$ ).

**Theorem 8.6.**

- (a) The set  $\{m_A \mid A \in \Xi_{n,d}^{ij}\}$  forms an  $\mathcal{A}$ -basis of  $\mathbf{S}_{n,d}^{ij}$ . Moreover, we have  $m_A \in [A] + \sum_{B \in \Xi_{n,d}^{ij}, B <_{\text{alg}} A} \mathcal{A}[B]$ .
- (b) We have a canonical basis  $\mathfrak{B}_{n,d}^{ij} := \{\{A\} \mid A \in \Xi_{n,d}^{ij}\}$  of  $\mathbf{S}_{n,d}^{ij}$  such that  $\overline{\{A\}} = \{A\}$  and  $\{A\} \in [A] + \sum_{B \in \Xi_{n,d}^{ij}, B <_{\text{alg}} A} v\mathbb{Z}[v][B]$ . Moreover, we have  $\mathfrak{B}_{n,d}^{ij} = \mathfrak{B}_{n,d}^c \cap \mathbf{S}_{n,d}^{ij}$ .

*Proof.* Part (a) is the counterpart of Theorem 5.18 (for  $\mathbf{S}_{n,d}^c$ ), and can be proved by the same argument, now with help from Theorem 8.4. The first half of Part (b) on the canonical basis follows by (a) and a standard argument (cf. [Lu93, 24.2.1]). The second half of (b) follows from the uniqueness characterization of the canonical basis  $\mathfrak{B}_{n,d}^{ij}$ .  $\square$

**8.3. Stabilization algebra of type  $ij$ .** We define two subsets of  $\tilde{\Xi}_n$  (6.1.1) as follows:

$$\tilde{\Xi}_n^{<} = \{A = (a_{ij}) \in \tilde{\Xi}_n \mid a_{00} < 0\}, \quad \tilde{\Xi}_n^{>} = \{A = (a_{ij}) \in \tilde{\Xi}_n \mid a_{00} > 0\}. \quad (8.3.1)$$

For any matrix  $A \in \tilde{\Xi}_n$  and  $p \in \mathbb{Z}$ , we define

$${}_p A = A + p(I - E^{00}). \quad (8.3.2)$$

**Lemma 8.7.** For  $A_1, A_2, \dots, A_f \in \tilde{\Xi}_n^{>}$ , there exists  $\mathcal{Z}_i \in \tilde{\Xi}_n^{>}$  and  $\mathcal{G}_i(v, v') \in \mathbb{Q}(v)[v', v'^{-1}]$  ( $i = 1, \dots, m$  for some  $m$ ) such that for all even integers  $p \gg 0$ , we have an identity in  $\mathbf{S}_{n,d}^c$  of the form:

$$[{}_p A_1] \cdot [{}_p A_2] \cdots [{}_p A_f] = \sum_{i=1}^m \mathcal{G}_i(v, v^{-p}) [{}_p \mathcal{Z}_i].$$

*Proof.* The proof is similar to the proof of Proposition 6.1 where  ${}_p A = A + pI$  is used instead of  ${}_p A$  (8.3.2). A detailed calculation shows that replacing  $p$  by  $\check{p}$  in (6.1.7) and (6.1.8) gives similar formulas, which proves the lemma for the base case ( $f = 2, A = A_2, B = A_1$  is tridiagonal). The lemma then follows by induction.  $\square$

Consequently, the vector space  $\dot{\mathbf{K}}_n^{c,>}$  over  $\mathbb{Q}(v)$  spanned by the symbols  $[A]$ , for  $A \in \tilde{\Xi}_n^{>}$ , is a stabilization algebra whose multiplicative structure is given by (with  $f = 2$ ; associativity follows from  $f = 3$ ):

$$[A_1] \cdot [A_2] \cdots [A_f] = \begin{cases} \sum_{i=1}^m \mathcal{G}_i(v, 1) [\mathcal{Z}_i] & \text{if } \text{col}_c(A_i) = \text{row}_c(A_{i+1}) \ \forall i, \\ 0 & \text{otherwise.} \end{cases} \quad (8.3.3)$$

By arguments entirely analogous to those for Proposition 6.2 and Theorem 6.7,  $\dot{\mathbf{K}}_n^{c,>}$  admits a (stabilizing) bar involution,  $\dot{\mathbf{K}}_n^{c,>}$  admits a monomial basis  $\{m_A \mid A \in \tilde{\Xi}_n^{>}\}$ , and a stably canonical basis  $\dot{\mathfrak{B}}^{c,>}$ .

**Definition 8.8.** Let  $\dot{\mathbf{K}}_n^{\mathfrak{v}}$  be the  $\mathbb{Q}(v)$ -submodule of  $\dot{\mathbf{K}}_n^{\mathfrak{c},>}$  with a standard basis  $\{[A] \mid A \in \tilde{\Xi}_n^{\mathfrak{v}}\}$ , where

$$\begin{aligned} \tilde{\Xi}_n^{\mathfrak{v}} &= \{A = (a_{ij}) \in \tilde{\Xi}_n \mid a_{0i} = a_{i0} = \delta_{0i}\} \\ &= \{A \in \tilde{\Xi}_n^> \mid \text{col}_c(A)_0 = \text{row}_c(A)_0 = 0\}. \end{aligned} \quad (8.3.4)$$

The goal of this subsection is to realize  $\dot{\mathbf{K}}_n^{\mathfrak{v}}$  as a subquotient of  $\dot{\mathbf{K}}_n^{\mathfrak{c}}$  with compatible bases by following [BKLW, Appendix A] (where an algebra  $\dot{\mathbf{U}}^i$  is realized as a subquotient of an algebra  $\dot{\mathbf{U}}^j$  with compatible stably canonical bases).

It follows by the second characterization for  $\tilde{\Xi}_n^{\mathfrak{v}}$  in (8.3.4) that  $\dot{\mathbf{K}}_n^{\mathfrak{v}}$  is a subalgebra of  $\dot{\mathbf{K}}_n^{\mathfrak{c},>}$ . Since the bar-involution on  $\dot{\mathbf{K}}_n^{\mathfrak{c},>}$  restricts to an involution on  $\dot{\mathbf{K}}_n^{\mathfrak{v}}$ , we reach the following conclusion.

**Lemma 8.9.** *The set  $\dot{\mathbf{K}}_n^{\mathfrak{v}} \cap \dot{\mathfrak{B}}^{\mathfrak{c},>}$  forms a stably canonical basis of  $\dot{\mathbf{K}}_n^{\mathfrak{v}}$ .*

The submodule of  $\dot{\mathbf{K}}_n^{\mathfrak{c}}$  spanned by  $[A]$  for  $A \in \tilde{\Xi}_n^{\mathfrak{v}}$  is not a subalgebra. This is why we need a somewhat different stabilization above to construct the stably canonical basis for  $\dot{\mathbf{K}}_n^{\mathfrak{v}}$ . We shall see below the stabilization above is related to the stabilization used earlier.

Define  $\mathbf{J}$  to be the  $\mathbb{Q}(v)$ -submodule of  $\dot{\mathbf{K}}_n^{\mathfrak{c}}$  spanned by  $[A]$  for all  $A \in \tilde{\Xi}_n^<$ .

**Lemma 8.10.** *The submodule  $\mathbf{J}$  is a two-sided ideal of  $\dot{\mathbf{K}}_n^{\mathfrak{c}}$ .*

*Proof.* Since  $\mathbf{J}$  is clearly invariant under the anti-involution  $[A] \mapsto v^{-d_A+d_{tA}}[{}^tA]$  for  $\dot{\mathbf{K}}_n^{\mathfrak{c}}$ , The claim that  $\mathbf{J}$  is left ideal of  $\dot{\mathbf{K}}_n^{\mathfrak{c}}$  is equivalent to that  $\mathcal{J}$  is a right ideal of  $\dot{\mathbf{K}}_n^{\mathfrak{c}}$ . We shall show that  $\mathbf{J}$  is left ideal of  $\dot{\mathbf{K}}_n^{\mathfrak{c}}$ . To that end, it suffices to show that  $[B][A] \in \mathbf{J}$  for arbitrary  $A \in \tilde{\Xi}_n^<$  and tridiagonal  $B \in \tilde{\Xi}_n$ .

By the multiplication formula, the matrices corresponding to the terms showing up in  $[B] \cdot [A]$  must be of the form

$$A^{(T-S)} = A - (T - S)_\theta + (\widehat{T - S})_\theta, \quad T \in \Theta_{B,A}, S \in \Gamma_T.$$

Suppose that the  $(0,0)$ -entry  $a_{00} - 2(t_{00} - s_{00}) + 2(\widehat{T - S})_{00}$  is positive. Note that we have

$$\begin{aligned} \llbracket A; S; T \rrbracket &= \prod_{(i,j) \in I_a^+} \left[ \begin{array}{c} (A - T_\theta)_{ij} + s_{ij} + s_{-i,-j} + (\widehat{T - S})_{ij} + (\widehat{T - S})_{-i,-j} \\ (A - T_\theta)_{ij}; s_{ij}; s_{-i,-j}; (\widehat{T - S})_{ij}; (\widehat{T - S})_{-i,-j} \end{array} \right] \\ &\quad \cdot \prod_{k \in \{0, r+1\}} \left( \frac{\prod_{i=1}^{s_{kk} + (\widehat{T - S})_{kk}} [a_{kk} - 2t_{kk} - 1 + 2i]}{[s_{kk}]![(\widehat{T - S})_{kk}]!} \right) \cdot \llbracket S \rrbracket. \end{aligned}$$

Hence  $\llbracket A; S; T \rrbracket = 0$  and the term  $[A^{(T-S)}]$  makes no contribution to  $[B][A]$ . Therefore, we have  $[B][A] \in \mathbf{J}$ .  $\square$

Recall from (6.4.1) a semi-monomial basis element  $m'_A = [A^{(1)}][A^{(2)}] \cdots [A^{(x)}]$ , and from (6.4.2) a monomial basis element  $m_A = \{A^{(1)}\}\{A^{(2)}\} \cdots \{A^{(x)}\}$ .

**Lemma 8.11.** *We have  $m'_A \in \mathbf{J}$ , for  $A \in \tilde{\Xi}_n^<$ .*

*Proof.* From the construction of  $A^{(i)}$  in  $m'_A = [A^{(1)}][A^{(2)}] \cdots [A^{(x)}]$ , the matrix  $A^{(x)}$  has the same diagonal entries as  $A$  and hence  $[A^{(x)}] \in \mathbf{J}$ . Thus it follows from Lemma 8.10 that  $m'_A \in \mathbf{J}$ .  $\square$

Recall  $\dot{\mathbf{K}}_n^c$  admits a stably canonical basis of  $\dot{\mathfrak{B}}^c$  from Theorem 6.7.

**Corollary 8.12.** *The ideal  $\mathbf{J}$  admits a monomial basis  $\{m_A \mid A \in \tilde{\Xi}_n^{\leq}\}$ , and a stably canonical basis  $\dot{\mathfrak{B}}^c \cap \mathbf{J} = \{\{A\} \mid A \in \tilde{\Xi}_n^{\leq}\}$ .*

*Proof.* Recall the new basis  $\mathbf{f}_A$  from [FLLLW, Section 9.4]. By arguing in a way identical to the proof of [BKLW, Lemma A.6], we obtain that the  $\mathbf{f}_A$  lies in  $\mathbf{J}$  for all  $A \in \tilde{\Xi}_n^{\leq}$  and hence forms a basis for  $\mathbf{J}$ . Note that in the proof the role of Lemma A.5 in *loc. cit.* is played by Lemma 8.10. By [FLLLW, Proposition 9.4.4], we see that  $\mathbf{f}_A$  is bar-invariant, and so is  $\mathbf{J}$ . As a consequence, the set of  $\{A\}$  for all  $A \in \tilde{\Xi}_n^{\leq}$  is a basis of  $\mathbf{J}$ , by an argument similar to [BKLW, Proposition A.7]. So Lemma 8.10 implies that the monomial  $m_A = \{A^{(1)}\}\{A^{(2)}\} \cdots \{A^{(x)}\}$  is in  $\mathbf{J}$ , for any  $A \in \tilde{\Xi}_n^{\leq}$ . The corollary follows.  $\square$

**Proposition 8.13.** *The following statements hold.*

- (a) *The quotient algebra  $\dot{\mathbf{K}}_n^c/\mathbf{J}$  admits a monomial basis  $\{m_A + \mathbf{J} \mid A \in \tilde{\Xi}_n^{>}\}$ .*
- (b) *The quotient algebra  $\dot{\mathbf{K}}_n^c/\mathbf{J}$  admits a stably canonical basis  $\{\{A\} + \mathbf{J} \mid A \in \tilde{\Xi}_n^{>}\}$ .*
- (c) *The map  $\sharp : \dot{\mathbf{K}}_n^c/\mathbf{J} \rightarrow \dot{\mathbf{K}}_n^{c,>}$  sending  $[A] + \mathbf{J} \mapsto [A]$  is an isomorphism of  $\mathbb{Q}(v)$ -algebras, which matches the corresponding monomial bases and stably canonical bases.*

*Proof.* Parts (a) and (b) follow directly from Corollary 8.12. For (c), since the map  $\sharp$  is clearly a linear isomorphism, we still need to show that  $\sharp$  is an algebra homomorphism. We write

$$([B] + \mathbf{J}) \cdot ([A] + \mathbf{J}) = \sum_C f_{B,A}^C ([C] + \mathbf{J}) \in \dot{\mathbf{K}}_n^c/\mathbf{J}, \quad [B] \cdot [A] = \sum_C g_{B,A}^C [C] \in \dot{\mathbf{K}}_n^{c,>}$$

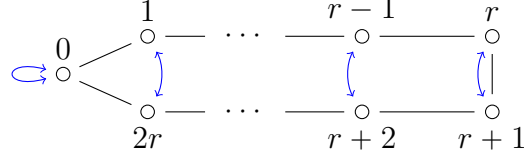
for some structure constants  $f_{B,A}^C$  and  $g_{B,A}^C$  with  $A, B, C \in \tilde{\Xi}_n^{>}$ . A detailed calculation shows that  $f_{B,A}^C = g_{B,A}^C$  when  $B$  is tridiagonal, and hence  $\sharp$  is an algebra homomorphism. Since  $\sharp$  matches the tridiagonal generators, it also matches the (semi-)monomial bases, and  $\sharp$  commutes with the bar involutions. Finally,  $\sharp$  matches the stably canonical bases as the partial orders on the two algebras are compatible.  $\square$

We summarize Lemma 8.9 and Proposition 8.13 above as follows.

**Theorem 8.14.** *As a  $\mathbb{Q}(v)$ -algebra,  $\dot{\mathbf{K}}_n^{v,j}$  is isomorphic to a subquotient of  $\dot{\mathbf{K}}_n^c$ , with compatible standard, monomial, and stably canonical bases.*

**Remark 8.15.** As in §7.2, §7.3 and §7.4, we can construct an algebra  $\mathbf{K}_n^{v,j}$  as a subalgebra of a completion of the algebra  $\dot{\mathbf{K}}_n^{v,j}$ . Similar to Theorem 7.8, the pair  $(\mathbf{K}_n, \mathbf{K}_n^{v,j})$  forms an affine quantum symmetric pair associated to the involution as depicted in Figure 2. We omit the details.

By definition and Remark 6.9,  $\Psi_d : \mathbf{K}_n^c \rightarrow \mathbb{Q}\mathbf{S}_{n,d}^c$  factors through  $\mathbf{J}$  and hence we obtain a surjective homomorphism  $\tilde{\Psi}_d : \mathbf{K}_n^c/\mathbf{J} \rightarrow \mathbb{Q}\mathbf{S}_{n,d}^c$ . We shall regard  $\mathbf{K}_n^{v,j} \subset \dot{\mathbf{K}}_n^{c,>}$  as a subalgebra of  $\mathbf{K}_n^c/\mathbf{J}$  via the identification  $\sharp : \dot{\mathbf{K}}_n^c/\mathbf{J} \cong \dot{\mathbf{K}}_n^{c,>}$  in Proposition 8.13.

FIGURE 2. Dynkin diagram of type  $A_{2r}^{(1)}$  with involution of type  $\iota_j$ .


Define  $\mathbb{Q}\mathbf{S}_{n,d}^{\iota_j} = \mathbb{Q}(v) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbf{S}_{n,d}^{\iota_j}$ . By restricting  $\tilde{\Psi}_d$  to  $\mathbf{K}_n^{\iota_j}$ , we obtain a surjective homomorphism  $\Psi_d^{\iota_j} : \mathbf{K}_n^{\iota_j} \rightarrow \mathbb{Q}\mathbf{S}_{n,d}^{\iota_j}$ . Putting all constructions together we have obtained the following commutative diagram

$$\begin{array}{ccc} \mathbf{K}_n^{\iota_j} & \longrightarrow & \mathbf{K}_n^c/\mathbf{J} \\ \Psi_d^{\iota_j} \downarrow & & \downarrow \tilde{\Psi}_d \\ \mathbb{Q}\mathbf{S}_{n,d}^{\iota_j} & \longrightarrow & \mathbb{Q}\mathbf{S}_{n,d}^c \end{array} \quad (8.3.5)$$

**Proposition 8.16.** (a) The homomorphism  $\Psi_d^{\iota_j} : \mathbf{K}_n^{\iota_j} \rightarrow \mathbb{Q}\mathbf{S}_{n,d}^{\iota_j}$  sends  $[A] \mapsto [A]$  for  $A \in \Xi_{n,d}^{\iota_j}$  and  $[A] \mapsto 0$  otherwise.

(b) The homomorphism  $\Psi_d^{\iota_j} : \mathbf{K}_n^{\iota_j} \rightarrow \mathbb{Q}\mathbf{S}_{n,d}^{\iota_j}$  preserves the (stably) canonical bases, that is, it sends  $\{A\} \mapsto \{A\}$  for  $A \in \Xi_{n,d}^{\iota_j}$  and  $\{A\} \mapsto 0$  otherwise.

*Proof.* By definition, the 2 horizontal maps and  $\tilde{\Psi}_d$  preserve the standard bases, and so by diagram chasing of (8.3.5) we conclude that  $\Psi_d^{\iota_j} : \mathbf{K}_n^{\iota_j} \rightarrow \mathbb{Q}\mathbf{S}_{n,d}^{\iota_j}$  preserves the standard basis, whence (a).

By Lemma 8.9 and Proposition 8.13, the top horizontal map preserves the stably canonical bases. The bottom horizontal map preserves the stably canonical bases by construction. The map  $\tilde{\Psi}_d$  preserves the (stably) canonical bases by Remark 6.9. Hence Part (b) follow by the same diagram chasing of (8.3.5).  $\square$

**8.4. Stabilization algebra of type  $j$ .** Recall  $\Xi_{n,d}$  from (2.3.5) and  $\Lambda = \Lambda_{r,d}$  from (2.2.1). Let

$$\begin{aligned} \Xi_{n,d}^j &= \{A \in \Xi_{n,d} \mid \text{row}_c(A)_{r+1} = 0 = \text{row}_c(A)_{r+1}\}, \\ \Lambda^j &= \{\lambda = (\lambda_i)_{i \in \mathbb{Z}} \in \Lambda \mid \lambda_{r+1} = 0\}, \\ \mathcal{D}_{n,d}^j &= \{(\lambda, g, \mu) \mid \lambda, \mu \in \Lambda^j, g \in \mathcal{D}_{\lambda\mu}\}. \end{aligned}$$

The following lemma is obtained by restriction of the bijection  $\kappa$  in Lemma 2.7 to  $\mathcal{D}_{n,d}^j$ .

**Lemma 8.17.** The map  $\kappa^j : \mathcal{D}_{n,d}^j \longrightarrow \Xi_{n,d}^j$  sending  $(\lambda, g, \mu)$  to  $(|R_i^\lambda \cap gR_j^\mu|)$  is a bijection.

Now we denote the affine  $q$ -Schur algebra of type  $j$  by

$$\mathbf{S}_{n,d}^j = \text{End}_{\mathbf{H}}\left(\bigoplus_{\lambda \in \Lambda^j} x_\lambda \mathbf{H}\right). \quad (8.4.1)$$

All results for  $\mathbf{S}_{n,d}^{\iota_j}$ ,  $\dot{\mathbf{K}}_n^{\iota_j}$  and  $\mathbf{K}_n^{\iota_j}$  in §8.1–8.3 admit counterparts in the current setting. We shall formulate the main statements but skip the details to avoid much repetition. The proposition below is an analogue of Theorem 8.6.

**Proposition 8.18.** *The algebra  $\mathbf{S}_{n,d}^{\mathfrak{n}}$  is naturally a subalgebra of  $\mathbf{S}_{n,d}^{\mathbf{c}}$ , admitting compatible standard, monomial and canonical bases.*

By repeating the constructions in §8.1–8.3 of the algebras of type  $\mathfrak{v}$ , we can obtain an associative algebra  $\dot{\mathbf{K}}_n^{\mathfrak{n}}$  with a standard basis  $[A]$  parametrized by

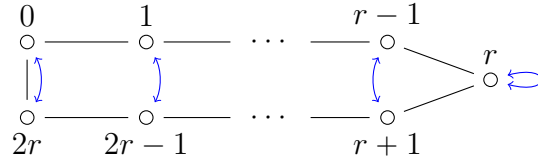
$$\begin{aligned} \tilde{\Xi}_n^{\mathfrak{n}} &= \{A = (a_{ij}) \in \tilde{\Xi}_n \mid a_{r+1,i} = a_{i,r+1} = \delta_{r+1,i}\} \\ &= \{A \in \tilde{\Xi}_n^> \mid \text{col}_{\mathbf{c}}(A)_{r+1} = \text{row}_{\mathbf{c}}(A)_{r+1} = 0\}. \end{aligned} \quad (8.4.2)$$

Similarly, we construct a subalgebra  $\mathbf{K}_n^{\mathfrak{n}}$  in a completion of  $\dot{\mathbf{K}}_n^{\mathfrak{n}}$ .

**Theorem 8.19.** *The following statements hold for  $\dot{\mathbf{K}}_n^{\mathfrak{n}}$ .*

- (a) *The algebra  $\dot{\mathbf{K}}_n^{\mathfrak{n}}$  admits a monomial basis and a stably canonical basis.*
- (b) *The algebra  $\dot{\mathbf{K}}_n^{\mathfrak{n}}$  is a subquotient of  $\dot{\mathbf{K}}_n^{\mathbf{c}}$  with compatible standard, monomial, and stably canonical bases.*
- (c) *The pair  $(\mathbf{K}_{n-1}, \mathbf{K}_n^{\mathfrak{n}})$  forms an affine quantum symmetric pair associated to the involution as depicted in Figure 3.*

FIGURE 3. Dynkin diagram of type  $A_{2r}^{(1)}$  with involution of type  $\mathfrak{n}$ .

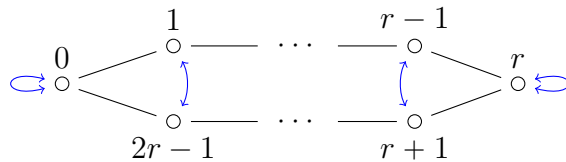


**Remark 8.20.** By Theorem 4.11 and a similar argument as that for [FL16, Theorem 4.8], one obtain an algebra isomorphism  $\mathbf{S}_{n,d}^{\mathfrak{v}} \simeq \mathbf{S}_{n,d}^{\mathfrak{n}}$ . Applying the stabilization process, one further establishes an algebra isomorphism  $\dot{\mathbf{K}}_n^{\mathfrak{v}} \cong \dot{\mathbf{K}}_n^{\mathfrak{n}}$ .

**8.5. Stabilization algebra of type  $\mathfrak{u}$ .** In the following we deal with the variant of affine Schur algebra of type  $\mathfrak{u}$  corresponding to the involution as depicted Figure 4. We set

$$\eta = n - 2 = \mathfrak{n} - 1.$$

FIGURE 4. Dynkin diagram of type  $A_{2r-1}^{(1)}$  with involution of type  $\mathfrak{u}$ .





Let

$$\Xi_{\eta,d}^u = \Xi_{n,d}^{uj} \cap \Xi_{n,d}^n, \quad \Lambda^u = \Lambda^n \cap \Lambda^j, \quad \mathcal{D}_{\eta,d}^u = \mathcal{D}_{n,d}^{uj} \cap \mathcal{D}_{n,d}^n.$$

The following lemma is obtained by restriction of the bijection  $\kappa$  in Lemma 2.7.

**Lemma 8.21.** *The map  $\kappa^u : \mathcal{D}_{\eta,d}^u \longrightarrow \Xi_{\eta,d}^u$  sending  $(\lambda, g, \mu)$  to  $(|R_i^\lambda \cap gR_j^\mu|)$  is a bijection.*

Now we define the *affine  $q$ -Schur algebra of type  $u$*  by

$$\mathbf{S}_{\eta,d}^u = \text{End}_{\mathbf{H}}\left(\bigoplus_{\lambda \in \Lambda^u} x_\lambda \mathbf{H}\right). \quad (8.5.1)$$

The algebra  $\mathbf{S}_{\eta,d}^u$  is naturally a subalgebra of  $\mathbf{S}_{n,d}^{uj}$ ,  $\mathbf{S}_{n,d}^n$  and  $\mathbf{S}_{n,d}^c$ , admitting compatible standard, monomial and canonical bases (similar to Proposition 8.18 for  $\mathbf{S}_{n,d}^n$ ).

By a similar process, we construct an associative algebra  $\dot{\mathbf{K}}_\eta^u$  with a standard basis  $[A]$  parametrized by

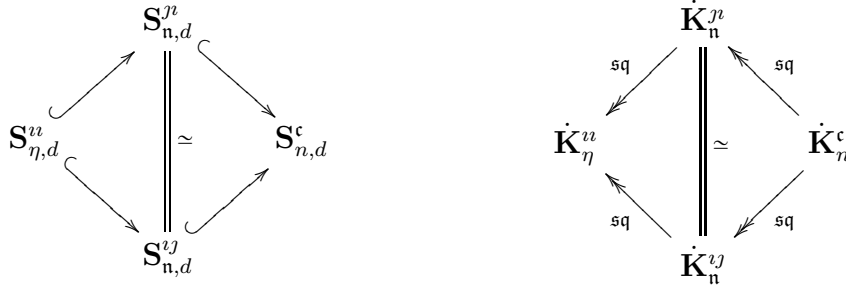
$$\tilde{\Xi}_\eta^u = \tilde{\Xi}_n^{uj} \cap \tilde{\Xi}_n^n. \quad (8.5.2)$$

We collect the main results for  $\dot{\mathbf{K}}_\eta^u$  which are similar to  $\dot{\mathbf{K}}_n^{uj}$  and  $\dot{\mathbf{K}}_n^n$  earlier (see Theorem 8.19) in the following theorem. The proofs are very similar to the previous cases, and hence skipped.

**Theorem 8.22.** *The following statements hold for  $\dot{\mathbf{K}}_\eta^u$ .*

- (a) *The algebra  $\dot{\mathbf{K}}_\eta^u$  admits a monomial basis and a stably canonical basis.*
- (b) *The algebra  $\dot{\mathbf{K}}_\eta^u$  is a subquotient of  $\dot{\mathbf{K}}_n^n$  (and of  $\dot{\mathbf{K}}_n^{uj}$ , respectively), with compatible standard, monomial, and stably canonical bases.*
- (c) *The pair  $(\mathbf{K}_\eta, \mathbf{K}_\eta^u)$  forms an affine quantum symmetric pair associated to the involution as depicted in Figure 4.*

The interrelation among Schur algebras as well as stabilization algebras of the four different types can be summarized below.



On the Schur algebra level, we have a commuting diagram for inclusions of Schur algebras. On the stabilization algebra level, we have the following commutative diagram of subquotients, where the notation  $\mathbf{K}_1 \xrightarrow{\text{sq}} \mathbf{K}_2$  stands for the statement that  $\mathbf{K}_2$  is a subquotient of  $\mathbf{K}_1$ . All the subquotients between various pairs of algebras preserve the stably canonical bases.

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